Complex Analysis

Math 312 — MWF 10:30-11:25am
Spring 1996 — 201 Fowler

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Worksheet #29

SUMMARY Implications of Cauchy's Integral Formula

READ page 130 (Section 43)

Implication of Cauchy's Integral Formula

Let Γ be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing Γ and z_0 is **any point inside** of Γ, then

$$\int_\Gamma \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

and

$$\int_\Gamma \frac{f(z)}{(z-z_0)^m}dz = \frac{2\pi i}{(m-1)!}f^{(m-1)}(z_0)$$

These two results lead to a number of other results. Actually, the two formulas are just restatement of one formula, known as the *generalized Cauchy Integral Formula.* Can you see how the first expression (**CIF**) is just a special case ($m=$??) of the second one?

Examples

We have rewritten the integral formulas in this way so that we can use them to actually evaluate integrals. Let's try these two.

$\oint_C \frac{e^{5z}}{z^3}\,dz =$

(where C is $|z|=1$ traversed once clockwise)

$\int_C \frac{2z+1}{z(z-1)^2}\,dz =$

(where C is given in the sketch)

The results we will be considering are

- **Cauchy's Inequality**
- **Liouville's Theorem**
- **Cauchy's Residue Theorem**

The first two can be derived from the generalised CIF almost immediately.

Cauchy's Inequality
Let f be analytic inside and on a circle $|z - z_0| = R$, taken in the positive sense, and denoted by C_R, then the derivatives of f as z_0 satisfy

$$|f^n(z_0)| \leq \frac{n!M}{R^n} \qquad (n = 1, 2, 3, \ldots)$$

where $|f(z)| \leq M$ for all z on C_R.

Proof: (Use the generalized CIF to try and bound $f^{(n)}(z_0)$)

Liouville's Theorem
If f is *entire* and *bounded* in the complex plane, then $f(z)$ is constant throughout the plane.

Proof: (Use Cauchy's Inequality when $n = 1$ and assume the radius has to be big enough to include the entire plane, e.g. $R \to \infty$)

Cauchy's Residue Theorem
If f is analytic on a simple (positively oriented) closed contour Γ and everywhere inside Γ except the finite number of points $z_1, z_2, \cdots z_n$ inside Γ, then

$$\oint_\Gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(z_k)$$

Before we talk about what a **residue** of a function $f(z)$ at z_k, denoted by $\text{Res}(f, z_k)$ or $\text{Res}(z_k)$ we will need to have a discussion about **poles**,**zeroes** and **singularities**.