Complex Analysis

Math 312 — MWF 10:30-11:25am

Spring 1996 — 201 Fowler

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Worksheet #26

SUMMARY Path Independence and the Cauchy-Goursat Theorem

READ pages 95-108 (Sections 33-35)

We have been practicing our complex integration skills. Today we will learn how to use **antiderivatives** and the **Cauchy-Goursat Theorem** to evaluate contour integrals more efficiently (simply!)

Example

Compute $\int_C 3z + i - 1 \; dz$ where C is contour from $-i$ to 1. Split into groups and each group should evaluate the integral using one of the following contours

- clockwise arc of $|z| = 1$ from $-i$ to 1
- counter-clockwise arc of $|z| = 1$ from $-i$ to 1
- direct line segment from $-i$ to 1
- contour consisting of two line segments, $-i$ to 0 and then 0 to 1
- contour consisting of two line segments, $-i$ to $1 - i$ and then $1 - i$ to 1

Using the results above, what do you think is the value of the contour integral $\int_C 2z^2 \; dz$, where C is any contour from $-i$ to i.

$\int_{-i}^{i} 2z^2 \; dz =$

Will it depend on which contour is used from $-i$ to i?

Recall that the value of $\int_C 2\overline{z}^2 \; dz$ **did** depend on the contour when we evaluated it in the previous class.

We need an explanation for this situation. How is $f(z) = z^2$ "like" $3z + i - 1$? How is $f(z) = z^2$ a different function than $g(z) = \overline{z}^2$?

Independence of Path

THEOREM: Suppose that the function $f(z)$ is continuous in a domain (open connected set) D and has an antiderivative $F(z)$ throughout D, i.e. $dF/dz = f(z)$ at each point in D. Then for *every* contour Γ lying in D connecting z_1 to z_2 we have the result

$$\int_\Gamma f(z)\,dz = F(z_2) - F(z_1)$$

In otherwords, the value of the integral is **independent of the path** chosen to link z_1 and z_2

So, getting back to the difference between $f(z) = z^2$ and $g(z) = \overline{z}^2$ we know that z^2 has the property that it is continuous and is equal to the derivative of $z^3/3$ on the open set $z \in \mathbf{C}$. $g(z)$ has no similar antiderivative.

Note that according to the above theorem the function $F(z)$ will be **analytic** and **continuous** on the domain D (since it has a derivative at every point of the open set D).

A corollary of the above theorem is the famous

Cauchy-Goursat Theorem: If $f(z)$ is analytic at all points interior to and on any simple closed contour Γ, then

$$\int_\Gamma f(z)dz = 0.$$

(Is this result surprising?)

Examples

1. $\displaystyle\int_C \sin z + 2e^z\,dz,$ where $C connects 0$ to πi

2. $\displaystyle\int_C \frac{dz}{z^2-4}\,dz$ where $C : |z| = 1$

QUESTION OF THE DAY (Due: 4/10)

Evaluate the following integrals

$$\int_{-i}^{i} \frac{dz}{z^2} \qquad \text{and} \qquad \int_{C_r} \frac{dz}{z^2} \text{ where } C_r : |z| = 1$$