Complex Analysis

Math 312 MWF 10:30-11:25am

Spring 1996 201 Fowler

Ron Buckmire ~ 402 Fowler ~ 259-2536 ~ ron@oxy.edu ~ Mon/Wed 1-3pm & Thu 1-2pm

Worksheet #24

SUMMARY Contours and Introduction to Complex Integration

READ pages 88-99 (Sections 31-33)

Smooth Arcs

An arc is said to be **smooth** if it obeys the following three conditions

- $z(t)$ has a CONTINUOUS DERIVATIVE on the interval $[a, b]$
- $z'(t)$ is not zero on (a, b)
- $z(t)$ is a one-to-one function on $[a, b]$

If the first two conditions are met but $z(a) = z(b)$, then it is called a **smooth closed curve**.

Examples

Give an example of a parametric representation of an arc and draw a sketch of a smooth arc in the complex plane below

Contours

A **contour** is a piecewise smooth arc. That is, $z(t)$ is continuous but $z'(t)$ is only piecewise continuous. If $z(a) = z(b)$ then it is called a *simple closed contour*.

Contours are important because they are the sets that **complex integration**, or integration of complex functions of a complex variable, are defined on.

Jordan curve theorem

A simple closed curve or simple closed contour divides the complex plane into two sets, the *interior* which is BOUNDED, and the *exterior*, which is UNBOUNDED.

Complex Integration

First, let's recall how to integrate complex functions of a **real** variable. Compute the following:

$$(a)\int_1^2 \frac{-i}{t^2} + (t+2i)^3\, dt \qquad\qquad (b)\int_0^\infty e^{-z^2 t}\, dt$$

Contour Integration

Integration of a complex function of a **complex** variable is performed on a set of connected points from, say, z_1 to z_2. It is a **contour integral**. Given a contour C defined as $z(t)$ for $a \le t \le b$ where $z_1 = z(a)$ and $z_2 = z(b)$, an integral of a complex function of a complex variable $f(z)$ is written as

$$\int_C f(z)dz \qquad \text{or} \qquad \int_{z_1}^{z_2} f(z)\, dz$$

Let $f(z)$ be piecewise continuous on $z(t)$. If C is a **contour** then $z'(t)$ is piecewise continuous on $a \le t \le b$ and we can redefine the integral of $f(z)$ along C as:

$$\int_C f(z)\, dz = \int_a^b f[z(t)]z'(t)\, dt$$

Example

Compute $\int_C \mathrm{Re}\, z\, dz$ where C is a directed line segment from $z = 0$ to $z = 1 + 2i$

1: write down a parametricization for C, $z(t)$

2: Convert the integral into an integral in t variable

3: Integrate!

QUESTION OF THE DAY

Show that

$$\int_{C_r} (z - z_0)^n\, dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1 \end{cases}$$

where n is any integer and C_r is a circle of radius r around z_0 (what is the equation of such a shape?) traversed **once** in the counter-clockwise direction