Complex Analysis

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Worksheet #22

SUMMARY Complex Functions of a **real** variable and complex contours

READ pages 85-93 (Sections 30-31)

Previously we have considered complex functions of a complex variable, such as $w = f(z)$ Now we want to consider complex functions which have a real variable as their argument. For example,

$$w(t) = u(t) + iv(t)$$

For the most part these functions just act like real functions. They consist of two real functions of one variable. They can be differentiated and integrated just like real functions.

Properties

$$\begin{aligned}
w'(t) &= u'(t) + iv'(t) \\
\int_a^b w(t)\ dt &= \int_a^b u(t)\ dt + i\int_a^b v(t)\ dt \\
\text{Re}\int_a^b w(t)\ dt &= \int_a^b \text{Re}(w(t))\ dt \\
\text{Im}\int_a^b w(t)\ dt &= \int_a^b \text{Im}(w(t))\ dt \\
\left|\int_a^b w(t)\ dt\right| &\leq \int_a^b |w(t)|\ dt
\end{aligned}$$

Examples

Consider $w_1(t) = 1 + it^2$ and $w_2 = e^{3it}$ Compute the following

1. $w_1' =$
2. $w_2' =$
3. $\displaystyle\int_0^2 w_1\ dt =$
4. $\displaystyle\int_0^2 w_2\ dt =$

Complex valued functions of a real variable are extremely useful in that they map a set of real points to a set of points in the complex plane.

Arcs

A point set $\gamma : z = (x, y)$ in the complex plane is said to be an **arc** if $x = x(t)$ and $y = y(t)$ where $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuos functions of t (which is real). The set γ is described by $z(t)$ where

$$z(t) = x(t) + i(y(t), \qquad a \leq t \leq b$$

The arc γ is said to be a **simple arc** (also called a *Jordan arc*) if the arc never crosses itself. However, if the curve would be simple except it crosses at the endpoints, i.e. $z(b) = z(a)$, it is known as a **simple closed curve** or *Jordan curve*.

Length

The length of an arc is given by

$$\mathcal{L} = \int_a^b |z'(t)| dt$$

Examples

Give an example of a parametric representation of an arc and draw a sketch of a different arc in the complex plane below

Smooth Arcs

An arc is said to be **smooth** if it obeys the following three conditions

- $z(t)$ has a CONTINUOUS DERIVATIVE on the interval $[a, b]$
- $z'(t)$ is not zero on (a, b)
- $z(t)$ is a one-to-one function on $[a, b]$

If the first two conditions are met but $z(a) = z(b)$, then it is called a **smooth closed curve**.

Contours

A **contour** is a piecewise smooth arc. That is, $z(t)$ is continuous but $z'(t)$ is only piecewise continuous. If $z(a) = z(b)$ then it is called a *simple closed contour*.

Contours are important because they are the sets that **complex integration**, or integration of complex functions of a complex variable, are defined on.

Jordan curve theorem

A simple closed curve or simple closed contour divides the complex plane into two sets, the *interior* which is BOUNDED, and the *exterior*, which is UNBOUNDED.