Complex Analysis

Math 312 MWF 10:30-11:25am
Spring 1996 201 Fowler

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Worksheet #21

SUMMARY $\exp z$ and logarithms and conformal mappings

READ pages 222-246 (Sections 68-74), pages 345-352 (Appendix 2)

Consider the mapping consisting of the exponential function. Let's think about where $w = e^z$ would map **horizontal** and **vertical** lines.

Examples

Let the vertical line be given by Re $z = \alpha$, in that case, the image of this line is $w = e^{\alpha} \cdot e^{iy}$, where y can be any real value. What does this set of points look like? Sketch how the point w changes as the pre-image point changes with y [Hint: does $|w|$ depend on the size of y?

Now, figure out where $w = e^z$ maps horizontal lines and sketch them on the axes above.

Putting this information together we can see that the set $0 < y < \pi, x \leq 0$ gets mapped to the interior of the semi-circle ($|z| < 1$, Im $z > 0$

How would we have to change our pre-image set so that $w = e^z$ would map the pre-image to the entire upper half-plane?

More mappings

Now we know that the exponential function maps vertical and horizontal line segments to portions of circles and rays. The logarithm function is the inverse of the exponential function so we can use this information to determine its mapping properties.

As usual, we have to determine which *branch* of the logarithm we are talking about. For most purposes, we will just be interested in the principal branch, Log z. I usually think the best way to think about the logarithm mapping is to try and think about the reverse of what the exponential function would do.

Here is an example of a useful mapping using Log z

The use of analytic functions to map one domain to another is very useful. Look at appendix 2 of the text to see further examples.

Conformal Mappings

Before we leave this topic of mappings we should put it in a general perspective. The type of mappings we have been looking at are called **conformal**.

All analytic functions are conformal at every point z_0 for which $f'(z_0) \neq 0$

We say that a mapping is **conformal** when it preserves the *angle* and *sense* between two directed curves as it maps them from the z plane to the w plane. If the magnitude of the angle but *not the sense* is preserved, the mapping is called **isogonal**

Another amazing property of conformal mappings is the fact that small areas ΔA_z get mapped to similarly-shaped areas ΔA_w and the scale factor of the transformation, also known as the **Jacobian** of the transformation, is given by $|f'(z)|^2$. Of course this only applies on the small scale of "infinitesimals".

The conformal nature of a mapping is a local property which tells you that this mapping will take a shape and make it "conform" to a new shape. An interesting global mapping property of analytic functions is that they are usually **open mappings.** That is, it maps open sets on to open sets.

Any analytic function which is nonconstant on a domain maps an open set to an open set.