# Complex Analysis

Math 312 — MWF 10:30-11:25am
Spring 1996 — 201 Fowler

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Worksheet #20

**SUMMARY** Finding and Using Linear Fractional Transformations

**READ** pages 216-220 (Sections 67-68)

**Linear Fractional Transformations (LFTs)**

We continue to consider transformations of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

Previously we showed that it took us evaluating $T(z)$ at 3 different points for us to determine exactly what the shape of the mapped image would look like.
In fact, to determine the equation of an LFT all you have to do is know where it maps 3 distinct points $z_1$, $z_2$ and $z_3$ to $w_1$, $w_2$ and $w_3$, where none of the points $z_i$ and $w_i$ is the point at infinity.
In that case the general form of the LFT is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

**Example**
Remember that in the previous class we dealt with an LFT which mapped took the interior of $|z-2| = 2$ to the negative complex plane. We saw that it mapped $0, 2+2i$ and $2$ to $0, -i/2$ and $-1/2$ (respectively). Using the formula above, show that the mapping that does this is

$$w = T(z) = \frac{z}{2z-8}$$

**Example**

Remember that we found out that the previous LFT mapped 4 to $\infty$. How do we determine the form of an LFT if one of the points is the point at infinity?

It actually makes the problem *easier*! Suppose we say that our LFT maps $z_2$ to $w_2 = \infty$. Simply let $w_2 = 1/\beta$ and take the limit as $\beta \to 0$ to show how the formula changes. Take 3 minutes and show that the general form of the LFT becomes

$$\frac{(w - w_1)}{(w - w_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

Try and write down the LFT that maps the points $\infty, i$ and $-1$ to $1, -i$ and $\infty$

One of the most frequently desired mappings is one which takes the upper-half of the extended complex plane to the interior of the unit circle. The text goes into the details of the derivation in section 67, but mappings which do it have the following form:

$$w = e^{i\alpha}\frac{z - z_0}{z - \overline{z_0}}, \qquad \text{Im } z_0 > 0, \alpha \in \Re$$

Using set notation (complex inequalities) write down a definition for the pre-image (upper half plane):

and the image (interior of the unit circle):

Draw a sketch illstrating the mapping below by drawing the $z$-plane and $w$-plane and give an example of a mapping which would produce the picture you drew.