Complex Analysis

Math 312 MWF 10:30-11:25am
Spring 1996 201 Fowler

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Worksheet #19

SUMMARY Möbius transformations (linear fractional transformations)

READ pages 213-220 (Sections 66-67)

Linear Fractional Transformations

Consider transformations of the form

$$w = f(z) = \frac{az+b}{cz+d}$$

They are also known as **bilinear transformations** or **Möbius transformations**. Show that you can re-write this to produce an expression of the form

$$Azw + Bz + Cw + D = 0$$

Find values for A, B, C and D in terms of a, b, c and d.

Notice that if $c = 0$ then our bi-linear transformation (linear in z and w) becomes just a linear transformation in z.
If $c \neq 0$ then we can re-write $w = f(z)$ as

$$w = \frac{a}{c} + \frac{bc - ad}{c}\frac{1}{cz+d}$$

If we look at the linear fractional transformation this way, we can see that it can be written as a composition of two linear transformations and an inverse mapping.

$$w = cz + d, \qquad w_1 = \frac{1}{w}, \qquad w_2 = \frac{a}{c} + \frac{bc - ad}{c}w_1$$

Find the composition of these three mappings above, so that $w_2 = T(z)$, and by so doing, show that T is a "LFT"

Therefore, we know that LFT's map *circles* and *lines* to ____________________ and ____________________

Properties of Linear Fractional Transformations

Let f be a Möbius transformation. Then

- f can be expressed as the composition of a finite number of rotations, translations, magnifications and inversions
- f maps the *extended complex-plane* to itself
- f maps the class of circles and lines to circles and lines
- f is **conformal** (i.e. $f'(z) \neq 0$) at every point besides its pole

A **pole** (regular singularity) of a function is a point z_0 where $\lim_{z \to z_0} f(z) = \infty$
Find the poles of

$$T(z) = \frac{az+b}{cz+d}$$

If a line or circle passes thru the pole of T then it must be mapped to a shape that goes thru the point at infinty. **Why?** What kind of shape would do that?

So, if a line or circle does NOT pass thru the pole of T it must get mapped to what kind of shape?

Where does T map the point at infinity to?

Since T is a one-to-one mapping on the extended complex plane, it has an inverse. If you solve $w = T(z)$ so that $z = T^{-1}(w)$, then

$$T^{-1}(w) = \frac{-dw+b}{cw-a}, \qquad (ad - bc \neq 0)$$

Note that T^{-1} is also an LFT. In general, if S and T are two LFTs, then$S[T(z)]$ is also an LFT.

Example

Find the image of the *interior* of the circle $|z-2| = 2$ under the LFT

$$w = f(z) = \frac{z}{2z-8}$$

Sketch the image and pre-image of C under $w = f(z)$