Complex Analysis

Math 312 — MWF 10:30-11:25am

Spring 1996 — 201 Fowler

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Worksheet #18

SUMMARY "Linear" transformations and the inversion transformation

READ pages 207-213 (Section 64-65) and pages 38-40 (Section 13)

Linear Transformation

Let us re-consider the idea that functions of a complex variable $w = f(z)$ represent a mapping from the complex z=(x,y) plane to the complex $w = (u, v)$ plane

Rotation: Rotation by an angle α

$w = f_1(z) = e^{i\alpha} z$

Scaling: Magnification/reduction by a factor of $|a|$

$w = f_2(z) = az$ (a is a real number)

Translation: Shifting by a vector of size (Re(B), Im(B))

$w = f_3(z) = z + B$

What do all all these transformations have in common?

Examples

Consider the circle $C_1 : |z - 1| = 1$. Find a series of transformations which maps C_1 onto $C_2 : |z - \frac{3}{2}i| = 2$

Sketch the series of transformations on the space below.

Using the relationship between *composition of functions* and *successive transformations* write down a single function which transforms C_1 into C_2

So, any **linear transformation** can be written as a composition of ____________________,
____________________ and ____________________

The Inversion Mapping $\frac{1}{z}$

The function $w = \frac{1}{z}$ establishes a one-to-one correspondence between the nonzero points of the z **and** w planes.
Remember $|z|^2 = z\overline{z}$, so $w = 1/z$ can be treated as two successive mappings

$$Z = \frac{1}{|z|^2} z, \qquad w = \overline{Z}$$

Think about the points

a. ($|z| > 1$): exterior to the circle $|z| = 1$

b. ($|z| < 1$): interior to the circle $|z| = 1$

c. ON the circle $|z| = 1$

Where do each of these sets get mapped to in the w-plane using the "inversion transformation?" [To answer this question you should pick a point in each one of these sets and see where it is mapped under the $w = 1/z$ transformation]

What about the point $z = 0$? What does "infinity" mean in the realm of complex numbers?

The Point at Infinity and The it Extended Complex Plane

We define the *extended complex plane* to consist of all points (Re z,Im z) plus ∞, which we call *the point at infinity.* The best way to think of the point of infinity is to think of a unit sphere centered at $z = 0$ being intersected by a plane along the equator. Each point P on the surface of the sphere corresponds to point z on the complex plane. The point N at the North Pole of the sphere corresponds to the point at infinity. The correspondence of the points z on the plane to the points P on the sphere is called *stereographic projection* and the sphere is known as the Riemann sphere
Now that we know about the point at infinity we can take the following limits

$$\lim_{z\to\infty} f(z) = w_0 \Longleftrightarrow \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$

PREGNANT PAUSE: Take 2 minutes and think about the concepts on this page.

Properties of the Inversion Mapping

The mapping $w = 1/z$ maps the extended complex plane to itself on a one-to-one basis.
The mapping $1/z$ transforms *circles and lines* into *circles and lines*

Lines passing thru the origin	$\longmapsto$	Lines passing thru the origin
Lines NOT passing thru the origin	$\longmapsto$	Circles passing thru the origin
Circles passing thru the origin	$\longmapsto$	Lines NOT passing thru the origin
Circles NOT passing thru the origin	$\longmapsto$	Circles NOT passing thru the origin