Complex Analysis

Math 312 MWF 10:30-11:25am
Spring 1996 201 Fowler

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Worksheet #15

SUMMARY More Elementary Functions: Complex Logarithms

READ Pages 74-80 (Sections 26 and 27)

Let us define $w = \log z$ as the inverse of $z = e^w$
But we know that $\exp[\ln|z| + i(\theta + 2n\pi)] = z$, where $n \in Z$, from our knowledge of the exponential function.
So we can define

$$\log z = \ln|z| + i \text{ arg } z = \ln|z| + i \text{ Arg } z + 2n\pi i$$

If we only use the principal value of the argument, then we define the principal value of $\log z$ as Log z, where

$$\text{Log } z = \ln|z| + i \text{ Arg } z$$

Examples
Compute $\text{Log }(-1), \text{Log }(1), \log(-1)$, and $\log(1)$

Identities
$z = e^{\log z}$ but $\log e^z = z + 2k\pi i$ (Is this a surprise?)
$\log(z_1 z_2) = \log z_1 + \log z_2$
$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
However these do not neccessarily apply to the principal branch of the logarithm, written as Log z.
Log z is a single-valued function and is analytic in the domain D^* consisting of all points of the complex plane *except for those lying on the nonpositive real axis*, where

$$\frac{d}{dz}\text{Log } z = \frac{1}{z}$$

Sketch the set D^* and convince yourself that it is an open connected set.

The set of points Re $z \leq 0$, Im $z = 0$ is a line of discontinuities known as a **branch cut**. By putting in a branch cut we say that we "construct Log z from $\log z$". Why can we not evaluate $\log z$ along the whole x-axis?

Branch

A single-valued function $F(z)$ is said to be a branch of a multiple-valued function $f(z)$ in a domain D if $F(z)$ is single-valued and continuous in D and has the property that for each $z \in D$, the value $F(z)$ is one of the values of $f(z)$

Examples

Determine the domain of analyticity for the function $f(z) = \text{Log }(3z - i)$ and compute $f'(z)$ What is $f(i)$? What about $f'(i)$?

One can define other analytic branches of $\log z$ by choosing different branch cuts.
The usual way to do this is to make the branch cut along $\theta = \alpha$ starting at the origin, so that

$$\log z = \ln |z| + i\theta, \qquad \text{where } \alpha < \theta < \alpha + 2\pi$$

These branches of $\log z$ can be denoted $\mathcal{L}_\alpha$, where $\theta = \alpha$ is the branch cut
A **branch point** of a function f is a point which is common to all branch cuts of f. So, 0 is a branch point of $\log z$

Example)

Find a branch of $\log(z^3 - 2)$ that is analytic at $z = 0$ and compute $f(0)$ and $f'(0)$

Roots of Complex Numbers (Reprise)

From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in Z$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right)$$

But

$$\begin{aligned} \exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln |z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in Z) \\ &= |z|^{1/n} \exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right) \\ &= |z|^{1/n} \exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z \end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where k is restricted to $0, 1, 2, \ldots, n-1$. Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when k can be any integer and n is fixed?]