Complex Analysis

Math 312 MWF 10:30-11:25am
Spring 1996 201 Fowler

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Worksheet #12

SUMMARY Analyticity and the Cauchy-Rieman Equations (CRE)

Analyticity
If the derivative $f'(z)$ exists at all points z of an *open set* G, then f is said to be **analytic** (or holomorphic or regular) on the set G.
If $f(z)$ is analytic on the whole complex plane, it is called **entire**.
If "f is analytic at the point z_0", what this really means is that f is analytic in a neigborhood of z_0.

When does the derivative of a function $f(z)$ exist? What if it is written in its component forms? Analyticity let's us answer these questions.

Analytic functions treat the variable z as a whole unit, so that when you are given two component parts $u(x,y)$ and $v(x,y)$ they can always be combined to form a complex function of the single variable $z = x + iy$.
Consider $f_1 = x^2 - y^2 + 2xyi$ and $f_2 = x^2 - y^2 + 3xyi$
By now you should be able to see that $f_1 = z^2$ while $f_2 = z^2 + i\text{Re}(z)\text{Im}(z)$
$f_1' = 2z$ while there is no derivative of f_2

Cauchy-Rieman Equations
Analyticity implies a relationship between the real $(u(x,y))$ and imaginary $(v(x,y))$ parts of a complex function $f(z)$. That relationship is known as the **Cauchy-Rieman Equations**, which we will abbreviate C.R.E.:

$$u_x = v_y, \qquad u_y = -v_x$$

Satisfying the CRE is considered to be a *necessary* condition for analyticity of a function, because if $f(z)$ is analytic then it is necessary that the CRE are also satisfied.

$$\text{ANALYTICITY} \Rightarrow \text{C.R.E.}$$

To make satisfying the CRE a *sufficient* condition one needs the added condition that the first derivatives of u and v are continuous. If both these conditions are true and f is defined on an open set, then f is analytic on the open set.

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = -i(u_y(x_0, y_0) + iv_y(x_0, y_0))$$

$$\text{ANALYTICITY} \Longleftrightarrow \text{C.R.E.} + \text{Continuity of } u_x, u_y, v_x, v_y$$

Example
Show that the function $f(z) = 1/z$ is analytic. To do that you will have to answer the following questions:

- What is its domain of definition?
- What are its component functions?
- Does it satisfy the CRE?
- Is it analytic?