Complex Analysis

Math 312 — MWF 10:30-11:25am

Spring 1996 — 201 Fowler

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Worksheet #10

SUMMARY Continuity/Differentiability of Complex Functions

Continuity

A complex function $f(z)$ is **continuous** at a point z_0 if *all three* of the following statements are true

- $\lim_{z \to z_0} f(z)$ exists
- $f(z_0)$ exists
- $\lim_{z \to z_0} f(z) = f(z_0)$

Consider the function below:

$$f(z) = \begin{cases} \dfrac{z^2+4}{z-2i} & z \neq 2i \\ \\ 3+4i & z = 2i \end{cases}$$

Answer the following questions

(a) What is the value of $\lim_{z \to 2i} f(z)$

(b) Is $f(z)$ continuous at $z = 2i$?

(c) Is $f(z)$ continuous at points $z \neq 2i$

We say that the function $f(z)$ defined above has a **removable singularity** at $z = 2i$. Write down the definition of $f(z)$ which has had the singularity removed.

Derivative

Let f be defined in a neigborhood around z_0. The **derivative** of f at z_0, denoted by $f'(z_0)$, is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

provided the above limit exists. The function f is said to be *differentiable* at z_0.

Consider $f(z) = |z|^2$. Write down the expression $\frac{\Delta w}{\Delta z}$
The derivative $\frac{dw}{dz} = f'(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Suppose we try and evaluate this limit for our function $f(z) = |z|^2$.
Write down the real and imaginary parts of the function $f(z) = |z|^2$

Write down the result when we use a vertical path along the complex axis. (sketch)

Write down the result when we use a horizontal path along the real axis. (sketch)

Which result is correct?
What does this tell you about the derivative of $f(z) = |z|^2$?
Write down $f'(z)$

The point of this example is that even though the component functions of a complex function may have an infinite number of continuous partial derivatives at a point, the complex function may not differentiable there.

DIFFERENTIABILITY $\Rightarrow$ CONTINUITY

CONTINUITY DOES NOT IMPLY DIFFERENTIABILITY.

Rules of Differentiation
The standard rules of differentiating function that you learned for real functions basically apply to complex functions. Scilicet:

$$\frac{d}{dz}(c) = 0 \qquad \frac{d}{dz}(z) = 1 \qquad \frac{d}{dz}(z^n) = nz^{n-1}$$

Linearity

$$\frac{d}{dz}(cf(z) + g(z)) = cf'(z) + g'(z) \qquad c \text{ constant}$$

Product Rule

$$\frac{d}{dz}[f(z)g(z)] = f'(z)g(z) + f(z)g'(z)$$

Quotient Rule

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$