
Complex Analysis

Math 312 Spring 2016

© 2016 Ron Buckmire

Fowler 309 MWF 11:45am-12:40pm

<http://sites.oxy.edu/ron/math/312/16/>

Class 28

TITLE Introduction to Linear Fractional Transformations

CURRENT READING Zill & Shanahan, §7.2

HOMEWORK SET #11 (DUE WED APR 20)

Zill & Shanahan, §7.2 14, 15, 21, **27***.

SUMMARY

We shall follow up on some odds and ends (Cauchy's Second Residue Theorem) and begin our in-depth look at the most useful mapping, the linear fractional transformation (LFT).

Cauchy's "Second Residue Theorem"

If a function $f(z)$ is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C , then

$$\oint f(z) dz = -2\pi i \mathbf{Res} \left[\frac{1}{z^2} f \left(\frac{1}{z} \right); \mathbf{0} \right]$$

In other words, instead of finding the residues of all the singularities of the given function $f(z)$ which lie inside the given contour C , all you need to do is find the residue at a single point, $z = 0$, of the associated function $\frac{1}{z^2} f \left(\frac{1}{z} \right)$. Note what's really going on involves finding the residue of the function at the point at infinity.

EXAMPLE

Evaluate $\oint_{|z|=2\pi} \tan(z) dz$ using Cauchy's Second Residue Theorem.

Evaluate $\oint_{|z|=2} \frac{e^{1/z}}{z-1} dz$ using Cauchy's Second Residue Theorem.

Exercise

Evaluate $\oint_{|z|=2} \frac{3z+2}{z^2+1} dz$ using Cauchy's Second Residue Theorem. (Confirm your answer by using Cauchy's (First) Residue Theorem.)

Recall The Properties of the Inversion Mapping

The mapping $w = 1/z$ maps the extended complex plane to itself on a one-to-one basis.

The mapping $w = \frac{1}{z}$ transforms *circles and lines* into *circles and lines*

Lines passing thru the origin	\mapsto	Lines passing thru the origin
Lines NOT passing thru the origin	\mapsto	Circles passing thru the origin
Circles passing thru the origin	\mapsto	Lines NOT passing thru the origin
Circles NOT passing thru the origin	\mapsto	Circles NOT passing thru the origin

Bilinear Transformations

Consider transformations of the form

$$w = f(z) = \frac{az + b}{cz + d} \quad \text{where } ad - bc \neq 0$$

They are also known as **linear fractional transformations** or **Möbius transformations**. It's easy to see that you can re-write this to produce an expression of the form

$$Azw + Bz + Cw + D = 0$$

where A, B, C and D can be expressed in terms of a, b, c and d .

Bilinear Transformation as composite mapping

Notice that if $c = 0$ then our bilinear transformation (linear in z and w) becomes just a linear transformation in z .

If $c \neq 0$ then we can re-write $w = f(z)$ as

$$w = \frac{a}{c} + \frac{bc - ad}{c} \frac{1}{cz + d}$$

To show this, all we have to do is remember polynomial division:

If we look at the linear fractional transformation this way, we can see that it can be written as a composition of two linear transformations and an inverse mapping.

$$w = cz + d, \quad w_1 = \frac{1}{w}, \quad w_2 = \frac{a}{c} + \frac{bc - ad}{c} w_1$$

Find the composition of these three mappings above, so that $w_2 = T(z)$, and by so doing, show that T is a "LFT."

Thus LFTs can be thought of as a _____ followed by a _____ followed by a _____.

Therefore, we know that LFT's map *circles* and *lines* to _____ and _____

Properties of Linear Fractional Transformations

Let f be a Möbius transformation. Then

- f can be expressed as the composition of a finite number of rotations, translations, magnifications and inversions
- f maps the *extended complex-plane* to itself
- f maps the class of circles and lines to circles and lines
- f is **conformal** (i.e. $f'(z) \neq 0$) at every point besides its pole

Poles and Fixed Points

A **pole** (regular singularity) of a function is a point z_0 where $\lim_{z \rightarrow z_0} f(z) = \infty$

A **fixed point** of a function $f(z)$ is a point z_0 such that $f(z_0) = z_0$. That is, a fixed point in the z -plane gets mapped to the same spot in the w -plane.

Find the poles of $T(z) = \frac{az+b}{cz+d}$. How many poles does it have? How many fixed points does it have? (These answers should depend on a, b, c and d .)

If a line or circle passes thru the pole of T then it must be mapped to a shape that goes thru the point at infinity. **Why?** What kind of shape would do that?

So, if a line or circle does NOT pass thru the pole of T it must get mapped to what kind of shape?

Where does T map the point at infinity?

THEOREM: Circle-Preserving Properts of LFTs

If C is a circle in the z -plane and if $w = T(z) = \frac{az+b}{cz+d}$ is an LFT, then the image of C under T is either a circle or a line in the extended complex w -plane. The image is a line if and only if $c \neq 0$ and the pole of T , $z = -d/c$ is on the circle C .

Question Does the function $w = \frac{2z-1}{iz+1}$ map the circle $|z-i| = 1$ to a circle or a line?

Answer _____

Inverses of LFTs

Since T is a one-to-one mapping on the extended complex plane, it has an inverse. If you solve $w = T(z)$ so that $z = T^{-1}(w)$, then

$$T^{-1}(w) = \frac{-dw + b}{cw - a}, \quad (ad - bc \neq 0)$$

Note that T^{-1} is also an LFT. So, the inverse of an LFT is another LFT. But, wait, there's more! In general, if S and T are two LFTs, then $S(T(z))$ is also an LFT, i.e. the composition of two LFTs is also an LFT.

EXAMPLE

Find the image of the *interior* of the circle $C : |z - 2| = 2$ under the LFT given by $w = f(z) = \frac{z}{2z - 8}$. Sketch the image and pre-image of C under $w = f(z)$.

Exercise

Show that the image of the unit disk $|z| \leq 1$ under the mapping $w = i\frac{z - 1}{z + 1}$ is the set $\{w \in \mathbb{C} : \text{Im}(w) \leq 0\}$.