Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

 — http://sites.oxy.edu/ron/math/312/16/

Class 26

TITLE Cauchy Principal Value of an Improper Integral and Evaluating Improper Integrals Using Contour Integration

CURRENT READING Zill & Shanahan, §6.6

HOMEWORK SET #10 (DUE WED APR 6)

Zill & Shanahan, §Chapter 5 Review 4,5,6,7,8,9,17. **38*,40***. §6.4 2,6,20. **25***.
§6.5 7, 2,12,17,23. §6.6 3,8,15,**24***.

SUMMARY

We can define an improper integral over all x values using something called a Cauchy Principal Value. Then we can use this idea to help us evaluate real improper integrals of the first kind (of even functions with certain conditions) using contour integrals and residues!

EXAMPLE

Consider the improper integral $\int_{-\infty}^{\infty} x^3 \, dx$

What do you have to do before you can evaluate the integral?

Is this the same value as $\lim_{R\to\infty} \int_{-R}^{R} x^3 \, dx$?

Cauchy Principal Value

The Cauchy Principal Value of an improper integral, denoted by p.v. $\int_{-\infty}^{\infty} f(x) \, dx$ is defined as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x) \, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x) \, dx$$

GROUPWORK

Find p.v. $\int_{-\infty}^{\infty} x \, dx$ and p.v. $\int_{-\infty}^{\infty} x^2 \, dx$

Compare these answers to the improper integral $\int_{-\infty}^{\infty} x \, dx$ and $\int_{-\infty}^{\infty} x^2 \, dx$

What's the difference between two sets of answers? Notice any patterns?

When The Cauchy Principal Value Equals The Improper Integral

The relationship between the Cauchy Principal Value of an improper integral and the improper integral can be sumarized as

convergence of $\int_{-\infty}^{\infty} f(x)\ dx$ IMPLIES p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ EXISTS

p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ EXISTS DOES NOT IMPLY convergence of $\int_{-\infty}^{\infty} f(x)\ dx$

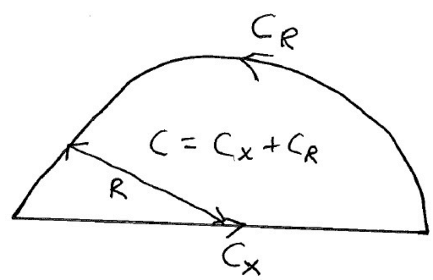
There is a condition on $f(x)$ from which we will know when the two values are equal: If $f(x)$ is an EVEN FUNCTION or if the improper integral converges.

p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ IS EQUAL TO $\int_{-\infty}^{\infty} f(x)\ dx$, when $f(x) = f(-x)$

Evaluation of real integrals of the form $\int_{-\infty}^{\infty} f(x)\ dx$ using Residues

We can also evaluate **real improper integrals** more easily by evaluating associated contour integrals. However, we have to have some boundedness conditions on the integrand $f(z)$ in order to do so.

Let C be a contour consisting of C_R [i.e. a semi-circular arc of radius R centered at the origin from $(R, 0)$ to $(-R, 0)$] combined with C_x [the horizontal linear path from $(-R, 0)$ to $(R, 0)$ along the x-axis].



$$\int_{C_x} f(z)\ dz + \int_{C_R} f(z)\ dz = \oint_C f(z)\ dz$$
$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = \oint_C f(z)\ dz$$

If we then take the limit as $R \to \infty$ and assume some boundedness conditions on $f(z)$ we can say that

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\ dx = \oint_C f(z)\ dz = 2\pi i \sum Res(f)$$ (NOTE: only residues in the upper half-plane)

(since if the boundedness theorems above apply to $f(z)$ then $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$)

`THEOREM`

Let $P(x)$ and $Q(x)$ be polynomials in x and let q the degree of $Q(x)$ exceed p the degree of $P(x)$ by at least 2, i.e. $q - p \geq 2$. Let $Q(x)$ be non-zero for all real values of x. Then

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_{\forall z_k \ni \text{Im}(z_k) > 0} \textbf{Res}\left(\frac{P(z)}{Q(z)}; z_k\right)$$

EXAMPLE

Show that $\displaystyle\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{\sqrt{2}}{4}\pi$

Exercise

Zill & Shanahan, page 318, #19 Show that $\displaystyle\int_{-\infty}^\infty \frac{1}{(x^2+1)^3}\,dx = \frac{3\pi}{8}$

Jordan's Lemma

THEOREM

Let $P(x)$ and $Q(x)$ be polynomials in x and let q the degree of $Q(x)$ exceed p the degree of $P(x)$ by at least 1, i.e. $q - p \geq 1$. Let $Q(x)$ be non-zero for all real values of x and v be a positive real number. Then

$$\lim_{R\to\infty} \oint_{C_R} \frac{P(z)}{Q(z)} e^{ivz}\ dz = 0$$

which means that

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{ivx}\ dx = 2\pi i \sum_{\forall z_k \ni \mathrm{Im}(z_k)>0} \mathbf{Res}\left(\frac{P(z)}{Q(z)} e^{ivz}; z_k\right)$$

We can re-write this result as

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(vx) dx = \mathrm{Re}\left\{2\pi i \sum_{\forall z_k \ni \mathrm{Im}(z_k)>0} \mathbf{Res}\left(\frac{P(z)}{Q(z)} e^{ivz}; z_k\right)\right\}$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(vx) dx = \mathrm{Im}\left\{2\pi i \sum_{\forall z_k \ni \mathrm{Im}(z_k)>0} \mathbf{Res}\left(\frac{P(z)}{Q(z)} e^{ivz}; z_k\right)\right\}$$

GROUPWORK

Show that $\displaystyle\int_0^{\infty} \frac{x\cos(x)}{x^2+9}\ dx = 0$ and $\displaystyle\int_0^{\infty} \frac{x\sin(x)}{x^2+9}\ dx = \frac{\pi}{2} e^{-3}$