Complex Analysis

Math 312 Spring 2016 | Fowler 309 MWF 11:45am-12:40pm

 | http://sites.oxy.edu/ron/math/312/16/

Class 25

TITLE Using Complex Integrals To Evaluate Real (Trigonometric) Integrals
CURRENT READING Zill & Shanahan, §6.6
HOMEWORK SET #10 (DUE WED APR 6)
Zill & Shanahan, §Chapter 5 Review 4,5,6,7,8,9,17. **38*,40***. §6.4 2,6,20. **25***.
§6.5 7, 2,12,17,23. §6.6 3,8,15,**24***.

SUMMARY
The beauty of Complex Residue Calculus is that it allows us to evaluate a vast number of contour integrals. In fact, we can show that we can use residues to evaluate associated **real** integrals which would otherwise be very difficult to get exact values for become quite easy as contour integrals. We can also use Cauchy's Residue Theorem to write down some infinite series as a sum of residues which we can evaluate and thus get closed form solutions for the value of an associated infinite series!

Recall the definition of $z = e^{i\theta} = \cos(\theta) + i\sin(\theta)$
Therefore, we can write $\cos(\theta)$ and $\sin(\theta)$ in terms of z.

$$\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right), \qquad \sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

We can use this information to convert integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta$ into contour integrals on $|z| = 1$.

EXAMPLE

Rewrite the integral $\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$ in terms of z, using $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$.

We can evaluate the real integral by evaluating the associated contour integral, $\oint_{|z|=1} \frac{2}{z^2+4iz-1}dz$

`FORMULA`

If a function $f(z) = h(z)/g(z)$ has a simple pole at z_0 then the residue can be computed by using the formula $\mathbf{Res}(f; z_0) = \frac{h(z_0)}{g'(z_0)}$

GROUPWORK

Show that $I = \int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta = \frac{i}{2} \oint_{|z|=1} \frac{z^6 + 1}{z^3(2z^2 - 5z + 2)} dz = \frac{\pi}{12}$

(HINT: $\cos 3\theta = \frac{e^{3i\theta} + e^{-3i\theta}}{2}$)

NOTE

$\mathbf{Res}\left(\frac{z^6+1}{z^3(2z^2-5z+2)}, 0\right) = \frac{21}{8}$; $\mathbf{Res}\left(\frac{z^6+1}{z^3(2z^2-5z+2)}, \frac{1}{2}\right) = -\frac{65}{24}$;

$\mathbf{Res}\left(\frac{z^6+1}{z^3(2z^2-5z+2)}, 2\right) = \frac{65}{24}$

Using Residues To Evaluate Infinite Series

Using $f(z) = \dfrac{\pi \cot(\pi z)}{p(z)}$ which has a finite number r poles at $z_{p_1}, z_{p_2}, \ldots, z_{p_r}$ where $p(z)$ has **(i)** real coefficients, **(ii)** degree $n \geq 2$ and **(iii)** no integer zeroes then

$$\sum_{k=-\infty}^{\infty} \frac{1}{p(k)} = -\sum_{j=1}^{r} \mathbf{Res}\left(\frac{\pi \cot(\pi z)}{p(z)}, z_{p_j}\right)$$

Using $g(z) = \dfrac{\pi \csc(\pi z)}{p(z)}$ where $p(z)$ has the same conditions as before, then

$$\sum_{k=-\infty}^{\infty} (-1)^k \frac{1}{p(k)} = -\sum_{j=1}^{r} \mathbf{Res}\left(\frac{\pi \csc(\pi z)}{p(z)}, z_{p_j}\right)$$

EXAMPLE

We'll obtain the result that $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{a}\coth(\pi a)$ when $p(z) = z^2 + a^2$.

NOTE: $\cot(ia) = -i\coth(a)$.

Exercise

Use the result $\displaystyle\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi}{a}\coth(\pi a)$ to show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \frac{\pi a\coth(\pi a) - 1}{2a^2}$

Then we can show that $\displaystyle\lim_{a\to 0^+} \sum_{k=1}^{\infty} \frac{1}{k^2+a^2} = \lim_{a\to 0^+} \frac{\pi a\coth(\pi a) - 1}{2a^2} \Rightarrow \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.
(Use L'Hopital's Rule!)

HINT: The Laurent Series Expansion for $\coth z \approx \dfrac{1}{z} + \dfrac{z}{3} - \dfrac{z^3}{45} + \ldots$ for $0 < |z| < \infty$