# Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

 http://sites.oxy.edu/ron/math/312/16/

## *Class 22*: Friday March 25

**TITLE** The Many, Many Implications of Cauchy's Integral Formula(s)
**CURRENT READING** Zill & Shanahan, §5.4-5.5
**HOMEWORK SET #10 (DUE WED APR 6)**
Zill & Shanahan, §Chapter 5 Review 4,5,6,7,8,9,17. **38*,40***. §6.4 2,6,20. **25***.
§6.5 7, 2,12,17,23. §6.6 3,8,15,**24***.

**SUMMARY**
Cauchy's Integral Formula leads to some of the most famous results in mathematics.

## Applications of Cauchy's Integral Formula

Let $C$ be a simple closed (positively oriented) contour. If $f$ is analytic in some simply connected domain $D$ containing $C$ and $z_0$ is **any point inside** of $C$, then

$$\oint_C \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

and

$$\oint_C \frac{f(z)}{(z-z_0)^m}\,dz = \frac{2\pi i}{(m-1)!} f^{(m-1)}(z_0)$$

These two results lead to a number of other results. Actually, the two formulas are just restatement of one formula, known as the *generalized Cauchy Integral Formula*. Can you see how the first expression (**CIF**) is just a special case ($m =$??) of the second one?

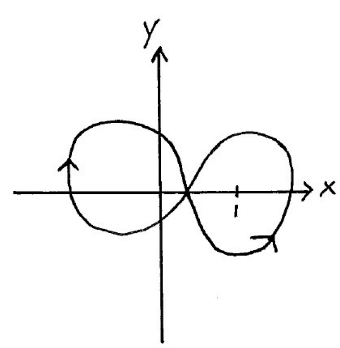
EXAMPLES

We have rewritten the integral formulas in the way above so that we can use them to actually evaluate integrals. Let's to do the following two.

$\displaystyle\oint_C \frac{e^{5z}}{(z-1)^3}\,dz =$

(where $C$ is $|z| = 2$ traversed once clockwise)

$\displaystyle\int_C \frac{2z+1}{z(z-1)^2}\,dz =$




(where $C$ is given in the sketch)

There are numerous theorems which directly follow from Cauchy's Integral Formula. I have listed a *few* of the more famous ones below...

# Implications of Cauchy's Integral Formula

### Morera's Theorem

If $f(z)$ is continuous in a simply-connected region $R$ and if $\oint_C f(z)dz = 0$ around *every* simple closed curve $C$ in $R$, then $f(z)$ is analytic in $R$.
(NOTE: Morera's Theorem is the converse of the Cauchy-Goursat theorem.)

### Cauchy's Inequality

If $f(z)$ is analytic inside and on a circle of radius $r$ and centered at $z = z_0$ then

$$|f^{(n)}(z_0)| \leq \frac{M \cdot n!}{r^n} \qquad n = 0, 1, 2, \ldots$$

where $M$ is an upper bound on $|f(z)|$ on $C$

### Liouville's Theorem

Suppose that for all $z$ in the entire complex plane, if $f(z)$ is analytic and bounded, (i.e. $|f(z)| < M$ for some real constant $M$) then $f(z)$ must be a constant.

### Fundamental Theorem of Algebra

Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has atleast one root.

### Gauss' mean value theorem

If $f(z)$ is analytic inside and on a circle $C$ with center $z_0$ and radius $r$ then $f(z_0)$ is the mean of the values of $f(z)$ on $C$, namely

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta$$

### Maximum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve $C$ and is not identically equal to a constant, then the maximum value of $|f(z)|$ occurs on $C$.

### Minimum modulus theorem

If $f(z)$ is analytic inside and on a simple closed curve $C$ and $f(z) \neq 0$ inside $C$, then the minimum value of $|f(z)|$ occurs on $C$.

### The Argument Theorem

Let $f(z)$ be analytic inside and on a simple closed curve $C$ except for a finite number of poles inside $C$. Then

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{f(z)} dz = N - P$$

where $N$ and $P$ are the number of zeroes and poles of $f(z)$ inside $C$

### Rouch$\acute{e}$ Theorem

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve $C$ and if $|g(z)| < |f(z)|$ on $C$, then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside of $C$.