Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

 — http://sites.oxy.edu/ron/math/312/16/

Class 18: Monday March 16

TITLE Introduction to Contour Integration
CURRENT READING Zill & Shanahan, §5.1 and §5.2;
HOMEWORK SET #8 (DUE WED MAR 23)
HOMEWORK Zill & Shanahan, §5.1 #6, 7, 8, 11, 27 **20,33***; §5.2 # 2, 7, 10, 21, 22,**29***

SUMMARY
We shall begin to consider integration of a complex function of a complex variable and do our very first contour integrals!

Exercise
First, let's recall how to integrate complex functions of a **real** variable. Compute the following:

$$(a)\int_1^2 \frac{-i}{t^2} + (t+2i)^3\,dt \qquad\qquad (b)\int_0^\infty e^{-z^2 t}\,dt$$

Contour Integration
Integration of a complex function of a **complex** variable is performed on a set of connected points from, say, z_1 to z_2. It is a **contour integral**. Given a contour C defined as $z(t)$ for $a \le t \le b$ where $z_1 = z(a)$ and $z_2 = z(b)$, an integral of a complex function of a complex variable $f(z)$ is written as

$$\int_C f(z)\;dz \qquad \text{or} \qquad \int_{z_1}^{z_2} f(z)\;dz$$

Let $f(z)$ be piecewise continuous on $z(t)$. If C is a **contour** then $z'(t)$ is piecewise continuous on $a \le t \le b$ and we can redefine the integral of $f(z)$ along C as:

$$\int_C f(z)\;dz = \int_a^b f[z(t)]z'(t)\;dt$$

EXAMPLE
Compute $\displaystyle\int_C \text{Im } z\;dz$ where C is a directed line segment from $z = 2$ to $z = 2i$

Properties of Contour Integrals

(i.e. Integrals of Complex Functions of a Complex Variable) Suppose the function f and g are continuous complex functions of a complex variable in a domain D and C is a (piecewise) smooth curve lying entire in D, then

i. $\displaystyle\int_C kf(z)\ dz = k\int_C f(z)\ dz$ where $k \in \mathbb{C}$

ii. $\displaystyle\int_C [f(z)+g(z)]\ dz = \int_C f(z)\ dz + \int_C g(z)\ dz$

iii. $\displaystyle\int_C f(z)\ dz = \int_{C_1} f(z)\ dz + \int_{C_2} f(z)\ dz$ where C could be formed from joining C_1 and C_2 end to end.

iv. $\displaystyle\int_{-C} f(z)\ dz = -\int_C f(z)\ dz$ where $-C$ has the opposite orientation of C

The textbook Zill & Shanahan refers to these properties as Theorem 5.2.2 (page 214).

THEOREM: **Bounding Theorem**

If f is a function of a complex variable that is continuous along a smooth curve C and if $|f(z)| \leq M$ for all z on C and L is the length of C then

$$\left|\int_C f(z)\ dx\right| \leq ML$$

Evaluation of Contour Integrals

Given $f(z) = u(x,y) + iv(x,y)$, we can write

$$\int_C f(z)\ dz = \int_C [u+iv][dx+idy] = \int_C udx - vdy + i\int_C vdx + udy$$

In other words, every complex contour integration can really be thought of as two line integrals in real variables involving the functions $u(x,y)$ and $v(x,y)$.

Algorithm for Evluating Contour Integrals

(The steps to be taken to complete the process of contour integration)

1: Write down a parametrization for the contour, $z(t)$

2: Convert the integral into an integral in (real) t variables by finding an expression for the integrand: $f(z(t))z'(t)$

3: Integrate!

Exercise

Compute $\displaystyle\int_C \operatorname{Im} z\ dz$ where C is a contour consisting of two linear segments starting at $z = 2$ to $z = 2i$ by way of the origin.

GroupWork

Compute $\displaystyle\int_C 2\overline{z}^2\,dz$ where C is a directed line segment from $z = 2$ to $z = -2$. (Sketch the contour and evaluate the integral.)

Also evaluate $\displaystyle\int_C 2\overline{z}^2\,dz$, this time using C being a counterclockwise circular arc from $z = 2$ to $z = -2$. (Sketch the contour and then evaluate the integral.)

Also evaluate $\displaystyle\int_C 2\overline{z}^2\,dz$, this time using C being a clockwise circular arc from $z = 2$ to $z = -2$. (Sketch the contour and then evaluate the integral.)

Discussion Question

Does the value of your contour integral depend on the contour (i.e. the path taken from (2,0) to (-2,0)?

EXAMPLE

Show that

$$\oint_{C_r} (z - z_0)^n \, dz = \begin{cases} 2\pi i & n = -1 \\ \\ 0 & n \neq -1 \end{cases}$$

where n is any integer and C_r is a circle of radius r around z_0 (what is the equation of such a shape?) traversed **once** in the counter-clockwise direction. How will our results change if we reverse the direction of travel along the contour (i.e. move in a clockwise direction)?

Exercise

Evaluate $\displaystyle\oint_{|z|=1} \frac{1}{z} \, dz$ where the contour is traversed once in a clockwise direction.