Complex Analysis

Math 312 Spring 2016 Fowler 309 MWF 11:45am-12:40pm

 http://sites.oxy.edu/ron/math/312/16/

Class 17: **Monday March 14**

TITLE Introduction to Complex Integration
CURRENT READING Zill & Shanahan, §5.1 and §5.2;
HOMEWORK SET #8 (DUE WED MAR 23)
Zill & Shanahan, §5.1 #6, 7, 8, 11, 27 **20,33***; §5.2 # 2, 7, 10, 21, 22 **29***.

SUMMARY
We shall begin to consider integration of a complex function of a real variable by reviewing line (or path) integrals from *Multivariable Calculus* and thinking about how to classify curves in 2-dimensional space as the first steps we need to embark on our journey to evaluate integrals of a complex function of a complex variable.

DEFINITION: path integral

Given a vector function $\vec{f}:\mathbb{R}^n \to \mathbb{R}^n$ and a path or curve γ in $\mathbb{R}^n$ given by $\vec{g}(t):\mathbb{R}\to\mathbb{R}^n$ for $a \le t \le b$ the path integral of $\vec{f}$ over γ is given by $\displaystyle\int_a^b \vec{f}(\vec{g}(t))\cdot\frac{d\vec{g}}{dt}\,dt = \int_\gamma \vec{f}\cdot d\vec{x}$

In *Complex Analysis* we will be restricting ourselves to $\mathbb{R}^2$ where $\vec{x}\in\mathbb{R}^2$ will consist of the Cartesian coordinates (x, y) that we know and love, except we will also have the added information that x and y always have the relationship that $z = x + iy$.

EXAMPLE

Let's evaluate the following integrals where C is the quarter-circle in the first quadrant defined by $x(t) = 4\cos t$, $y(t) = 4\sin t$ for $0 \le t \le \pi/2$.

(a) $\displaystyle\int_C xy^2\,dx$ **(b)** $\displaystyle\int_C xy^2\,dy$ **(c)** $\displaystyle\int_C xy^2\,ds$

Exercise

Evaluate $\displaystyle\int_C xy\,dx + x^2\,dy$ where C is the graph of $y = x^3$ from $-1 \le x \le 2$.

We are going to connect our understanding of line or path integrals in $\mathbb{R}^2$ to contour integrals in the complex plane.

Curves in the Complex Plane

Arcs

A point set $\gamma : z = (x, y)$ in the complex plane is said to be an **arc** or **curve** if $x = x(t)$ and $y = y(t)$ where $a \leq t \leq b$, where $x(t)$ and $y(t)$ are continuous functions of t (NOTE: x, y and t are all real variables, NOT complex variables). The set γ is described by $z(t)$ where

$$z(t) = x(t) + iy(t), \qquad a \leq t \leq b$$

The point $z(a)$ is called **the initial point** of γ and the point $z(b)$ is called **the terminal point** of γ.

Simple Curves

The arc γ is said to be **simple** or a simple arc (also called a *Jordan arc*) if the arc never crosses itself (except possibly at its endpoints). Mathematically, this means that $z(t_1) \neq z(t_2) \Leftrightarrow t_1 \neq t_2$. However, if the curve would be simple except that it crosses at the endpoints, i.e. the initial point equals the terminal point or $z(b) = z(a)$, then this kind of curve is called a **simple closed curve** or *Jordan curve*.

Smooth Curves

A curve (or arc) is said to be **smooth** if it obeys the following three conditions

1. $z(t)$ has a CONTINUOUS DERIVATIVE on the interval $[a, b]$
2. $z'(t)$ is never zero on (a, b)
3. $z(t)$ is a one-to-one function on $[a, b]$

If the first two conditions are met but $z(a) = z(b)$, then it is called a **smooth closed curve**.

Contours

A **contour** is a piecewise smooth curve. That is, $z(t)$ is continuous but $z'(t)$ is only piecewise continuous. If $z(a) = z(b)$ then it is called a *simple closed contour*. Contours are important because they are the sets that **complex integration**, or integration of complex functions of a complex variable, are defined on.

Positive Orientation

The direction of increasing values in the real parameter t corresponds to the **positive direction** on a contour C. If the contour is closed the positive direction corresponds to the **counter-clockwise direction** or the direction in which you would walk so that the interior of the closed contour is always on your left.

Length of an Arc

The length of an arc is given by

$$L = \int_a^b |z'(t)| dt = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

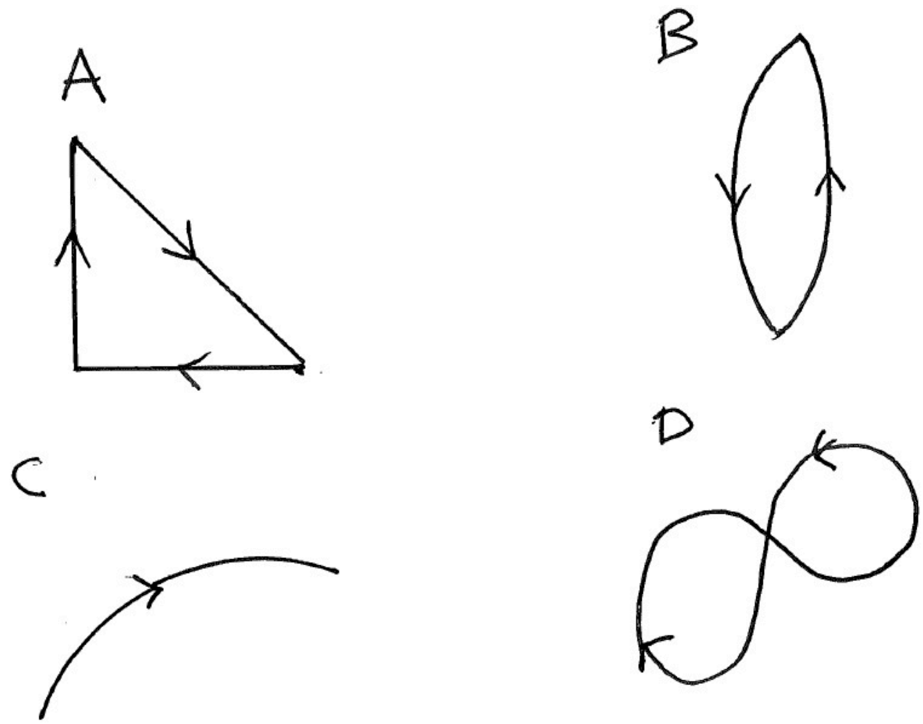
$x(t)$ and $y(t)$ can be thought of as parametric representations of the curve γ which consists of a set of points in the cartesian (x, y) plane.

THEOREM: **Jordan Curve Theorem**

A simple closed curve or simple closed contour divides the complex plane into two sets, the *interior* which is BOUNDED, and the *exterior*, which is UNBOUNDED.

GroupWork

Consider the following pictures of curves in the complex plane. Classify the orientation of the curves as positive or negative and determine which of the curves are smooth, simple, closed.



Which of these curves are:
simple?
smooth?
closed?
positively oriented?
contours?

Exercise

1. Write down a parametrization for a circular arc C_1 from $z = 2$ to $z = 2i$.

2. Write down a parametrization for C_2 which consists of two linear segments: one that starts at $z = 2$ but then travels horizontally to the origin and the second travels vertically from the origin ending at $z = 2i$

3. Sketch your contours C_1 and C_2

Complex Functions of a Real Variable

Now we want to consider complex functions which have a real variable as their argument. For example,

$$w(t) = u(t) + iv(t)$$

For the most part we can deal with these functions just like real functions. They consist of two real functions of one variable. They can be differentiated and integrated just like real functions of a real variable.

Properties of Integrals of Complex Functions of a real variable

$$\begin{aligned}
w'(t) &= u'(t) + iv'(t) \\
\int_a^b w(t)\ dt &= \int_a^b u(t)\ dt + i\int_a^b v(t)\ dt \\
\mathrm{Re}\int_a^b w(t)\ dt &= \int_a^b \mathrm{Re}(w(t))\ dt \\
\mathrm{Im}\int_a^b w(t)\ dt &= \int_a^b \mathrm{Im}(w(t))\ dt \\
\left|\int_a^b w(t)\ dt\right| &\leq \int_a^b |w(t)|\ dt
\end{aligned}$$

Exercise

Consider $w_1(t) = 1 + it^2$ and $w_2 = e^{3it}$ and answer the following

1. $w_1'(t) =$

2. $w_2'(t) =$

3. $\displaystyle\int_0^{2\pi} w_1\ dt =$

4. $\displaystyle\int_0^{2\pi} w_2\ dt =$

5. Draw sketches of w_1 and w_2 in the complex plane for $0 \leq t \leq 2\pi$

Complex valued functions of a real variable are extremely useful in that they map a set of real points to a set of points in the complex plane. These are exactly the same kinds of functions we have been using to describe some boundaries of sets in the Complex Plane in order to map their image under action by a complex function. YOU will need to grow even more familiar with (and comfortable using) these **parameterizations** in order to do integration in the complex plane.