Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

 — http://sites.oxy.edu/ron/math/312/16/

Class 15: Wednesday February 24

TITLE Complex Exponents z^c and c^z
CURRENT READING Zill & Shanahan, Section 4.2
HOMEWORK SET #7 (DUE WED MAR 16)
Zill & Shanahan, §4.5 9*,10*. Chapter 4 Review: 1, 2, 3, 9, 12, 14, 25, 26, **28***.

SUMMARY
We have previously looked at roots of complex numbers, i.e. complex numbers being raised to fractional exponents. Let's expand our thinking by considering complex numbers being raised to complex exponents!

Roots of Complex Numbers (Reprise)

From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in \mathbb{Z}$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right)$$

But

$$\begin{aligned} \exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln|z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in \mathbb{Z}) \\ &= |z|^{1/n}\exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right) \\ &= |z|^{1/n}\exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z \end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where k is restricted to $0, 1, 2, \ldots, n-1$. Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when k can be any integer and n is fixed?] What do you think happens if we try and raise a complex number to something besides an integer or rational number? How will we deal with **complex exponents**?

Complex Exponents

If $z \neq 0$ and $c \in \mathbb{C}$, the function z^c is defined as

$$z^c = \exp(\log z^c) = \exp(c \log z)$$

Since $\log z$ is a multi-valued function, z^c will have multiple values. How many values depends on the nature of c.

$$z^c = \begin{cases} z^{n/m} & \text{if } c \text{ is rational, i.e. } n/m & \text{finite number of values } (m) \\ z^n & \text{if } c = n, \text{ where } n \text{ is an integer} & \text{single value} \\ z^c & \text{all other complex numbers} & \text{infinite number of values} \end{cases}$$

EXAMPLE

Show that i^i is purely real.

GroupWork

Compute the following:

(a) $(0.5 - \frac{\sqrt{3}}{2}i)^3 =$

(b) $(-1)^{2/3} =$

(c) $(1+i)^{1-i} =$

Derivatives of z^c and c^z

If you choose a branch of z^c which is analytic on an open set, then

$$\frac{d}{dz}(z^c) = cz^{c-1} \tag{1}$$

where the branch of the log used in evaluating z^c is the same branch used in evaluating z^{c-1}.

Differentiation Rule For A Complex Power z^c

In order to consider *the complex power function* z^c an analytic function, we need to consider

$$z^c = e^{\log(z^c)} = e^{c\log(z)} = e^{c(\ln r + i\theta)}, \text{ where } -\pi < \theta < \pi, \text{ and } r = |z| > 0$$

EXAMPLE

We can then use the chain rule to prove the result in (1), i.e. $(z^c)' = cz^{c-1}$.

Differentiation Rule For A Complex Exponential c^z

Similarly, we can define the *complex exponential function with base c*

$$c^z = \exp(z \log c)$$

When a single value of $\log c$ is chosen, i.e. $\text{Log } c = \ln|c| + i\theta$ where $-\pi < \theta < \pi$ and $\theta = \text{Arg}(c)$ then c^z is an **entire function** such that

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\exp(z \text{ Log } c) = c^z \text{ Log } c \tag{2}$$

Exercise

You should be able to use the chain rule to prove the result in (2), i.e. $(c^z)' = c^z \text{ Log } c$.

EXAMPLE

If $f(z) = (1+i)^z$, Find $f'(1-i)$ (HINT: Use Ln)

GroupWork

Example 3 on Page 169 of Zill & Shanahan.
Find the derivative of the principal value of z^i at the point $z = 1 + i$.