Complex Analysis

Math 312 Spring 2016 — Fowler 309 MWF 11:45am-12:40pm

 — http://sites.oxy.edu/ron/math/312/16/

Class 14: Monday February 22

TITLE The Complex Logarithm
CURRENT READING Zill & Shanahan, Section 4.1
HOMEWORK SET #6 (DUE WED MAR 16)
Zill & Shanahan, §4.1.1: 3,4,7,8, **17***; §4.1.2: 23,31,34,42 **44***;
Zill & Shanahan, §4.2: 4,9,10,17, **13***; 4.3: 2,9,37 **53***;

SUMMARY
We shall return to the murky world of branch cuts as we expand our repertoire of complex functions to encounter the complex logarithm function.

The Complex Logarithm $\log z$

Let us define $w = \log z$ as the inverse of $z = e^w$.

NOTE

Your textbook (Zill & Shanahan) uses $\ln$ **instead of** $\log$ **and Ln instead of Log .**

We know that $\exp[\ln|z| + i(\theta + 2n\pi)] = z$, where $n \in \mathbb{Z}$, from our knowledge of the exponential function.

So we can define

$$\log z = \ln|z| + i \text{ arg } z = \ln|z| + i \text{ Arg } z + 2n\pi i = \ln r + i\theta$$

where $r = |z|$ as usual, and $\theta = \arg(z)$ is the argument of z. This is significant because it means that the complex logarithm function $\log z$ is a ____________________ function.

If we only use the principal value of the argument, then we define the principal value of $\log z$ as $\text{Log } z$, where

$$\text{Log } z = \ln|z| + i \text{ Arg } z = \text{ Log } |z| + i \text{ Arg } z$$

Exercise

Compute $\text{Log }(-2)$ and $\log(-2)$, $\text{Log }(2i)$, and $\log(2i)$, $\text{Log }(4)$ and $\log(4)$

Logarithmic Identities

$z = e^{\log z}$ but $\log e^z = z + 2k\pi i$ (Is this a surprise?)

$\log(z_1 z_2) = \log z_1 + \log z_2$

$\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$

However these do not necessarily apply to the principal value of the logarithm, written as $\text{Log } z$.

QUESTION Is $\text{Log }(-2) + \text{ Log }(-2) = \text{ Log }(4)$? Is Log(2) + Log(-2) = Log(-4)?

Log z: the Principal Branch of $\log z$

Log z is a single-valued function and is analytic in the domain D^* consisting of all points of the complex plane *except for those lying on the nonpositive real axis*, where

$$\frac{d}{dz}\text{ Log } z = \frac{1}{z}$$

Exercise

Sketch the set D^* in the space below convince yourself that it is an open connected set. (Ask yourself: Is every point in the set an interior point?) **Describe $D*$ using set notation.**

The set of points $\{z \in \mathbb{C} : \text{ Re } z \leq 0 \cap \text{ Im } z = 0\}$ is a line of discontinuities known as a **branch cut**. By putting in a branch cut we say that we "construct Log z from $\log z$."

`NOTE`

The **principal <u>BRANCH</u>** of $\log z$ is defined to be equal to $\ln|z| + i\theta$ where $-\pi < \theta < \pi$. The **principal <u>VALUE</u>** of $\log z$ is defined to be equal to $\ln|z| + i\theta$ where $-\pi < \theta \leq \pi$.

The Inverse Function of e^z Is The Principal Value of $\log z$

Recall from your past experience with the real exponential function $f(x) = e^x$, $-\infty << x < \infty$ and the real natural logarithm function $g(x) = \ln(x)$, $x > 0$ that they are inverses of each other. This means that $f(g(x)) = x$ and $g(f(x)) = x$.

Similarly, the inverse of the complex exponential function $f(z) = e^z$ is the **principal value** of the complex logarithm function.

EXAMPLE

Let's confirm that the inverse function of the complex exponential function $f(z) = e^z$ (where $z \in \mathbb{C}$) is $g(z) = \text{ Log } (z)$ (where $|z| > 0$ and $-\pi < \text{ Arg } z \leq \pi$), the principal value of the complex logarithm function.

Analyticity of Log z

We can use a version of the Cauchy-Riemann Equations in polar coordinates to help us investigate the analyticity of Log z

Q: Why don't we investigate the analyticity of $\log z$?
A: Because $\log(z)$ is not a single-valued function!
If $x = r\cos\theta$ and $y = r\sin\theta$ one can rewrite $f(z) = u(x, y) + iv(x, y)$ into $f = u(r, \theta) + iv(r, \theta)$ in that case, the CREs become:

$$u_r = \frac{1}{r} v_\theta, \qquad v_r = -\frac{1}{r} u_\theta$$

and the expression for the derivative $f'(z) = u_x + iv_x$ can be re-written

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

Exercise

Using this information, show that Log z is analytic and that $\frac{d}{dz}$ Log $z = \frac{1}{z}$.
(HINT: You will need to write down $u(r, \theta)$ and $v(r, \theta)$ for Log z)

Log z As A Mapping Function

Circular Arcs Get Mapped To Vertical Line Segments

$w =$ Log z maps the set $\{|z| = r \cap \theta_0 \leq \text{ Arg } z \leq \theta_1\}$ onto the vertical line segment $\{\text{ Re }(w) = \text{Ln}(r) \cap \theta_0 \leq \text{ Im }(w) \leq \theta_1\}$

Segments of Rays Get Mapped To Horizontal Line Segments

$w =$ Log z maps the set $\{r_0 \leq |z| \leq_1 \cap \text{Arg } z = \theta\}$ onto the horizontal line segment $\{\text{Ln}(r_0) \leq \text{ Re }(w) \leq \text{Ln}(r_1) \cap \text{ Im }(w) = \theta_1\}$

Complex Plane Without Origin Gets Mapped To Infinite Horizontal Strip Of Width 2π

$w =$ Log z maps the set $|z| > 0$ onto the region $\{-\infty < \text{ Re }(w) < \infty \cap -\pi < \text{ Im }(w) \leq \pi\}$

GroupWork

Find the image of the annulus $2 \leq |z| \leq 4$ under the mapping of the principal logarithm Log z.

Zill & Shanahan, page 164, #35. Solve the equation $e^{z-1} = -ie^3$.