
Complex Analysis

Math 312 Spring 2016

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Fowler 309 MWF 11:45am-12:40pm

<http://sites.oxy.edu/ron/math/312/16/>

Class 9: Monday February 8

TITLE Limits and Continuity of Complex Functions

CURRENT READING Zill & Shanahan, Section 3.1

HOMEWORK SET #4 (DUE WED FEB 17)

Zill & Shanahan, Chap 2 Review 1-10, §3.1.1: #2, 11, 17, **20***; §3.1.2: #28, 31, 37, **50***;

SUMMARY

We shall formally define the definition of the limit of a complex function to a point and use this definition to define the concept of **continuity** in the context of a complex function of a complex variable. We also introduce the idea of using branch cuts to create continuous functions.

Limits

Suppose that $f(z)$ is defined on a deleted neighborhood of $z_0 \in \mathbb{C}$. In order to say that $\lim_{z \rightarrow z_0} f(z) = w_0$ we must be able to show that

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni 0 < |z - z_0| < \delta \Rightarrow |f(z) - w_0| < \epsilon$$

This may look like dense mathematical language, but in english this means that for every positive real number ϵ (no matter how small) there exists a real number δ (which depends on the choice of ϵ) so that regardless of how close you get to the point z_0 in the deleted neighborhood around it in the z -plane you can also get arbitrarily close to the value w_0 in the w -plane.

Visualize The Limit

Draw a picture of the definition of the limit.

Prove The Limit

Let's try and prove the result $\lim_{z \rightarrow i} z^2 = -1$ (See Example 2 on page 100 of Zill & Shanahan)

Nonexistence of a Complex Limit

If $f(z)$ approaches two complex numbers $L_1 \neq L_2$ along two different paths towards z_0 then $\lim_{z \rightarrow z_0} f(z)$ does not exist.

Exercise

Show that $\lim_{z \rightarrow 0} \frac{z}{\bar{z}}$ does not exist. (HINT: pick a vertical path and a horizontal path)

Rules for Limits

The rules on limits of complex functions are identical to the rules for limits of real functions of real variables (as you'd expect)

Suppose that $\lim_{z \rightarrow z_0} f(z) = w_0$ and $\lim_{z \rightarrow z_0} F(z) = W_0$ then

$$\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0 \quad (1)$$

$$\lim_{z \rightarrow z_0} [f(z)F(z)] = w_0W_0 \quad (2)$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \quad (W_0 \neq 0) \quad (3)$$

$$\lim_{z \rightarrow z_0} |f(z)| = |w_0| \quad (4)$$

$$\lim_{z \rightarrow z_0} c = c \quad (5)$$

$$\lim_{z \rightarrow z_0} z^n = z_0^n \quad (6)$$

$$\lim_{z \rightarrow z_0} P(z) = P(z_0) \quad (\text{where } P(z) \text{ is a polynomial}) \quad (7)$$

THEOREM

Given $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$ and $w_0 = u_0 + iv_0$
 $\lim_{z \rightarrow z_0} f(z) = w_0$ **if and only if** $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$

GROUPWORK

Use the above properties to evaluate the following limits (note which properties you use).

(a) $\lim_{z \rightarrow 1+2i} 2|z| + iz^2 + 2.5 - i =$

(b) $\lim_{z \rightarrow 3\pi i} ze^z =$

(c) $\lim_{z \rightarrow 0} \frac{z^8 + z^4 + z^2 + z - 1}{z^3 + 4z^3 - 9} =$

(d) $\lim_{z \rightarrow 1-i} 2xy - ix^2 + iy^2 =$

Continuity

A complex function $f(z)$ is **continuous** at a point z_0 if *all three* of the following statements are true

- 1: $\lim_{z \rightarrow z_0} f(z)$ exists
- 2: $f(z_0)$ exists
- 3: $\lim_{z \rightarrow z_0} f(z) = f(z_0)$

Consider the function below:

$$f(z) = \begin{cases} \frac{z^2 + 4}{z - 2i}, & z \neq 2i \\ 3 + 4i, & z = 2i \end{cases}$$

Answer the following questions

1. What is the value of $\lim_{z \rightarrow 2i} f(z)$?
2. Is $f(z)$ continuous at $z = 2i$?
3. Is $f(z)$ continuous at points $z \neq 2i$?

We say that the function $f(z)$ defined above has a **removable singularity** at $z = 2i$.

Exercise

Write down a definition of $f(z)$ which is continuous, i.e. a $f(z)$ which has had the singularity removed.

More Aspects of Continuity

As with real functions of a real variable, **sums, differences, products and compositions** of continuous functions are continuous.

THEOREM

Complex polynomial functions are continuous on the entire complex plane. Functions with this property are often called **entire** functions.

THEOREM

$f(z)$ continuous $\iff u(x, y)$ and $v(x, y)$ continuous.

THEOREM

When $f(z)$ is continuous in a region R , then $|f(z)|$ is also continuous in the region R and if R is a *bounded and closed* set (i.e. it is **compact**) then there exists a positive number M so that $|f(z)| \leq M \quad \forall z \in R$.

Introduction to Branch Cuts and Branch Points

Recall the principal square root function $z^{1/2}$. This function is defined as $\sqrt{|z|}\exp\left(\frac{i\text{Arg}(z)}{2}\right)$. It has as its domain the set $\mathcal{D} = \{z \in \mathbb{C} : \mathbb{C} \setminus \{0\}\}$.

EXAMPLE

Show that the principal square root function is not continuous at $z = -1$.

If we remove the set of points along the negative real axis from \mathcal{D} (the domain of the principal square root function) and define a new function $f_1(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $-\pi < \theta < \pi$ this new function is called the **principal branch** of the multiple-valued square root function $z^{1/2}$. The set of points we removed, $\{z \in \mathbb{C} : \text{Im}(z) = 0 \cap \text{Re}(z) \leq 0\}$, is called a **branch cut** of $z^{1/2}$.

Q: How is the principal branch $f_1(z)$ different from the principal square root function?

A: The principal branch is continuous on its entire domain.

Q: Can we define other branches of $z^{1/2}$?

A: Yes, we could define another branch of the principal square root function as $f_2(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $\pi < \theta < 3\pi$. And a third branch of $z^{1/2}$ as $f_3(z) = \sqrt{|z|}e^{i\frac{\theta}{2}}$ where $0 < \theta < 2\pi$

Exercise

Show that $f_2(z) = -f_1(z)$ for all $|z| > 0$.

Branch of a Multivalued Function

A branch of a multi-valued function is a single-valued analogue which is continuous on its domain.

Branch Cut

The set of points that have to be removed from the domain of a multivalued function to produce a branch of the function.

Branch Point

The point in the complex plane which lies in every branch cut of a complex function. It is often the origin.