Complex Analysis

Math 312 Spring 2016 Fowler 309 MWF 11:45am-12:40pm
 http://sites.oxy.edu/ron/math/312/16/

Class 6: **Monday February 1**

TITLE Functions of a Complex Variable
CURRENT READING Zill & Shanahan, Section 2.1-2.2
HOMEWORK Zill & Shanahan, §2.1: #3, 8, 14, 20, 36, **27***; §2.2: 7, 11, 12, 22, **27***

SUMMARY
We expand our exploration of complex variables to start considering functions of a complex variable. One of the first things you'll notice is the immediate geometric interpretations of function evaluation.

Functions of a Complex Variable

Given a set S of complex numbers, a function f is a rule which assigns to each $z \in S$ a complex number w. The set S is known as the **domain of definition** of f. (NOTE: The "domain of definition" is not necessarily a **domain** in the formal mathematical sense of the word we discussed earlier.) The set of all $w \in \mathbb{C}$ given by $w = f(S)$ is called the **image of** S **under** f or the **range of** f and is often denoted S'.
The value $w = f(z)$ can be written as $u + iv$, in other words:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of two real variables. We can really think of a complex function as two associated functions of two variables. (This is the main reason why *Multivariable Calculus* is the pre-requisite for this class!)

EXAMPLE 1

Write $f(z) = z^2 - z + 2i$ in the form $w = u(x, y) + iv(x, y)$

In addition, given a complex function $w(x, y)$ you can always write it in terms of z, $\overline{z}$ and constants.

Exercise 1

Write $w(x, y) = x^2 + iy^2$ in terms of z and $\overline{z}$. (HINT: recall Quiz #1)

Real Function of a Complex Variable

It is possible to have a function of a complex variable which only produces real values. Some examples of such functions are ______________, ______________ and ______________.

Complex Functions As Mappings

As usual, operations using complex variables have geometric significance.
First, let's get more practice evaluating functions of a complex variable:
Using $f(z) = z^3$, compute

(a) $f(2) =$

(b) $f(\sqrt{2} + i\sqrt{2}) = f(2e^{i\pi/4})$

(c) $f(2i) =$

We can't really graph a complex function (**Why not?**), so what we usually do instead is show what the image of a complex function is on particular sets of points in the complex plane.

Complex Function of a (Single) Real Variable

Suppose $x(t)$ and $y(t)$ are functions of a real variable t. The set of points $\mathcal{D}$ consisting of all points $z(t) = x(t)+iy(t)$ for $a \leq t \leq b$ is called a **parametric curve in the complex plane** or a **complex parametric curve**. The function $z(t)$ is also called the **parametrization** of the curve $\mathcal{D}$ in the plane.

Common Parametric Curves in the Complex Plane

Line Segment (from z_0 to z_1)

$$z(t) = z_0(1 - t) + z_1 t, \qquad 0 \leq t \leq 1$$

Ray (emanating from z_0 at angle α to the horizontal axis)

$$z(t) = z_0 + te^{i\alpha}, \qquad 0 \leq t \leq \infty$$

Circle (centered at z_0 with radius r)

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi$$

Image of a Parametric Curve under a Complex Mapping

Given a complex function of a complex variable $w = f(z)$ and a complex parametric curve $\mathcal{D}$ represented by $z(t)$ for $a \leq t \leq b$ the function $w(t) = f(z(t))$ for $a \leq t \leq b$ is the parametrization of $\mathcal{D}'$, the image of $\mathcal{D}$.

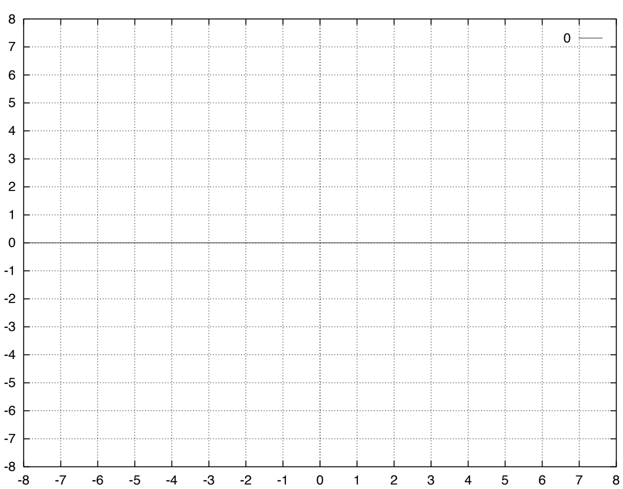
EXAMPLE 2

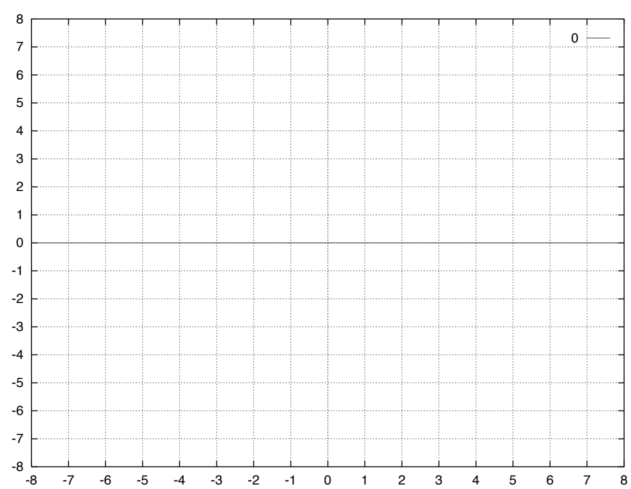
Example 3, ZS pg. 58. Let's find the image of the line segement from 1 to i under the mapping $w = \overline{iz}$.

GroupWork

If you consider $w = f(z)$ a mapping from the z-plane to the w-plane, find the image $\mathcal{D}'$ of $\mathcal{D}$ (the "quarter-disc" of radius 2 centered at the origin) in the first quadrant of the z-plane under the mapping $f(z) = z^3$.

1. Write down a definition of the "quarter disk of radius 2" using complex inequalities
2. Shade in this region on your (x, y) axes (z-plane) below (to the left)
3. Find a parametrization for a ray $\mathcal{A}$ emanating from the origin that lies in the quarter-disk.
4. Sketch $\mathcal{A}$ in the z-plane (to the left)
5. Find a parametrization for the image $\mathcal{A}'$ of your curve $\mathcal{D}$ under the mapping $f(z) = z^3$.
6. Sketch $\mathcal{A}'$ in the w-plane (to the right). Write down a description of this curve.
7. Use your knowledge of the image of a single curve in the region of interest to determine the image of the entire region. Shade in the mapped region on your (u, v) axes (w-plane) below (to the right)

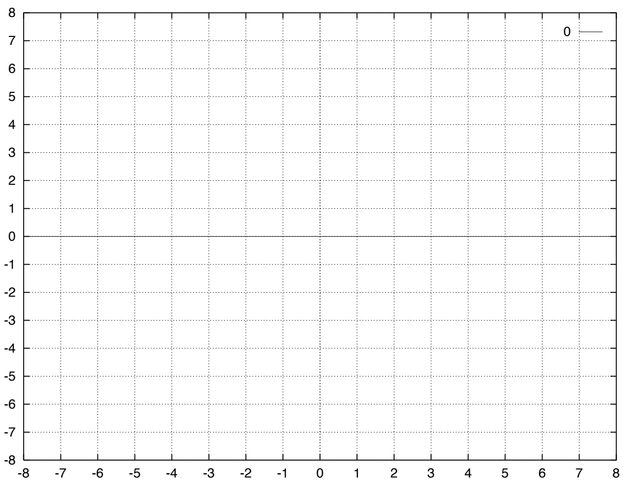


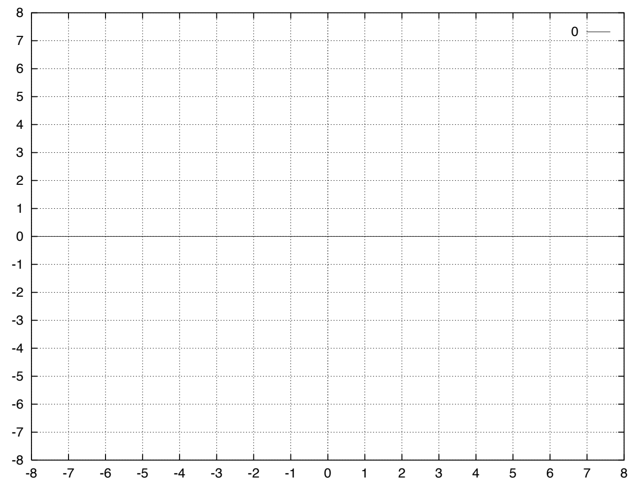


GroupWork

On the following set of axes of axes, do the following:

1. Find a parametrization for a curved arc $\mathcal{B}$ that lies in $\mathcal{D}$.
2. Sketch $\mathcal{B}$ in the z-plane (to the left)
3. Find a parametrization for the image $\mathcal{B}'$ of your curve $\mathcal{B}$ under the mapping $f(z) = z^3$.
4. Sketch $\mathcal{B}'$ in the w-plane (to the right)
5. Use your knowledge of the image of a single curve in the region of interest to determine the image of the entire region of interest under the given mapping. Shade in the mapped region $\mathcal{D}'$ on your (u, v) axes (w-plane) below (to the right)





Exercise

On the following set of axes, sketch the pre-image and the the image of the mapping of the unit "quarter-disk" under $f(z) = 2z^4 - 2 - i$ looks like in the w-plane. Use set notation to describe the image.