1. (10 points) We want to find a formula for an entire function \( f(z) \) but all we know is that its real part is given by \( u(x, y) = x^3 - 3xy^2 - 4xy + 6 \) and that it maps the point \((1, 1)\) to the origin.

(a) (6 points) Use the Cauchy-Riemann Equations (i.e. \( u_x = v_y, \ u_y = -v_x \)) to find the imaginary part of \( f(z) \), sometimes written as \( v(x, y) \), exactly.

\[ u_x = 3x^2 - 3y^2 - 4y = v_y \quad \text{(CRE)} \]

\[ u_y = 0 - 6xy - 4x + 0 = -u_x = 6xy + 4x \]

\[ 6xy + A'(x) = 6xy + 4x \]

\[ A'(x) = 4x \]

\[ A(x) = 2x^2 + C \]

\[ u = 3x^2 - y^2 - 2y^2 + 2x^2 + C \]

\[ u(1,1) = 0 = 1 - 1 - 4 + 6 = 0 \]

\[ v(1,1) = 0 = 3 - 1 - 2 + 2 + C \Rightarrow C = -2 \]

\[ f = u + iv = x^3 - 3xy^2 - 4xy + 6 + i(3x^2 - 3y^2 + 2x^2 - 2) \]

(b) (4 points) Confirm that both \( v(x, y) \) and its harmonic conjugate \( u(x, y) \) are examples of functions of two variables \( \phi(x, y) \) that satisfy the 2-dimensional Laplace Equation \( \phi_{xx} + \phi_{yy} = 0 \).

\[ u = x^3 - 3xy^2 - 4xy + 6 \]

\[ u_x = 3x^2 - 3y^2 - 4y \]

\[ u_{xx} = 6x \]

\[ u_y = -6xy - 4x \]

\[ u_{yy} = -6x \]

\[ u_{xx} + u_{yy} = 6x - 6x = 0 \]

\[ v = 3x^2y - y^3 - 2y^2 \]

\[ v_x = 6xy + 4x \]

\[ v_{xx} = 6y + 4 \]

\[ v_y = 3x^2 - 3y^2 - 4y \]

\[ v_{yy} = -6y - 4 \]

\[ v_{xx} + v_{yy} = 6y - 6y \]

\[ = 0 \]