Dirichlet in the Upper-Half Plane

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Overview

1. Evaluating real improper integrals
2. Using results to solve Dirichlet problem in upper-half plane

a. Real Improper Integrals

Use the techniques presented in Section 6.6 to establish the integral formulas

$$\int_{-\infty}^{\infty} \frac{\cos(s)}{s^2 + a^2} ds = \frac{\pi e^{-a}}{a}$$

$$\int_{-\infty}^{\infty} \frac{\sin(s)}{s^2 + a^2} ds = 0$$

To minimize confusion, we will make the substitution $s = x$

$$\int_{-\infty}^{\infty} \frac{\cos(s)}{s^2 + a^2} ds \to \int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx$$

$$\int_{-\infty}^{\infty} \frac{\sin(s)}{s^2 + a^2} ds \to \int_{-\infty}^{\infty} \frac{\sin(x)}{x^2 + a^2} dx$$

a. Real Improper Integrals

Referencing back to Euler's Formula, we see that:

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx + i\int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx$$

$$\downarrow$$

$$f(x) = \frac{1}{x^2 + a^2}, \quad \alpha = 1$$

$$\downarrow$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} e^{ix} dx = \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \cos x dx + i\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} \sin x dx$$

a. Real Improper Integrals

We can evaluate this real integral by considering the complex contour integral

$$\oint_C f(z)e^{iz}dz, \quad f(z) = \frac{1}{z^2 + a^2}$$

The contour C consists of $[-\infty, \infty]$ on the x-axis and the semicircle C_R in the upper half plane

$$\oint_C \frac{1}{z^2 + a^2}e^{iz}dz = \int_{C_R} \frac{1}{z^2 + a^2}e^{iz}dz + \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2}e^{ix}dx$$

a. Real Improper Integrals

From Cauchy's Residue Theorem, we know that

$$\oint_C f(z)dz = 2\pi i \sum Res(f(z), z_o)$$

For the contour integral we're considering:

$$\oint_C \frac{1}{z^2 + a^2} e^{iz} dz = 2\pi i \sum Res(f(z)e^{iz}, z_o)$$

The roots are $z_o = \pm\, ia$, both of order 1, but we only need to consider $z_o = ia$:

$$Res(f(z)e^{iz}, ia) = \frac{g(ia)}{h'(ia)} = \left.\frac{e^{iz}}{2z}\right|_{z=ia} = \frac{e^{-a}}{2ia}$$

$$\oint_C \frac{1}{z^2 + a^2} e^{iz} dz = 2\pi i (\frac{e^{-a}}{2ia}) = \frac{\pi e^{-a}}{a}$$

a. Real Improper Integrals

Using our result from the previous slide, we get

$$\int_{C_R} \frac{1}{z^2 + a^2} e^{iz} dz + \int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} e^{ix} dx = \frac{\pi e^{-a}}{a}$$

From Theorem 6.6.2, the integral over the semicircle becomes zero as R$\to \infty$, which leaves us with only the real portion

$$\int_{-\infty}^{\infty} \frac{1}{x^2 + a^2} e^{ix} dx = \frac{\pi e^{-a}}{a}$$

a. Real Improper Integrals

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2} \cos x dx + i \int_{-\infty}^{\infty} \frac{1}{x^2+a^2} \sin x dx = \frac{\pi e^{-a}}{a} + i0$$

Substituting back in s for x, we find the results to the initial integrals:

$$\boxed{\int_{-\infty}^{\infty} \frac{\cos s}{s^2+a^2} ds = \frac{\pi e^{-a}}{a} \quad and \quad \int_{-\infty}^{\infty} \frac{\sin s}{s^2+a^2} ds = 0}$$

b. Dirichlet Problem

Solve the Dirichlet Problem in the upper half-plane $y > 0$ subject to the boundary condition $\phi(x, 0) = f(x) = cos(x)$, $-\infty < x < \infty$. [Hint: Make the substitution $s = t - x$ and use the formulas in part (a).]

b. Dirichlet Problem

Theorem (Poisson Integral Formula for the Half-Plane)

If $f(x)$ is a piece-wise continuous and bounded function on $-\infty < x < \infty$, then the solution to the Dirichlet problem in the upper half-plane $y > 0$ with boundary condition $\phi(x, y) = f(x)$ at all points of continuity of f is given by

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2} dt$$

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Note: $(x-t)^2 = (t-x)^2$. Our integral can now be written as

$$\phi(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\cos(t)}{(t-x)^2+y^2}dt$$

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$$\downarrow$$

$$\phi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\cos(s+x)}{s^2 + y^2} ds$$

b. Dirichlet Problem

Using the trig identity

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We showed in (a) that

$$\int_{-\infty}^{\infty} \frac{\cos(s)}{s^2 + a^2} ds = \frac{\pi e^{-a}}{a} \quad and \quad \int_{-\infty}^{\infty} \frac{\sin(s)}{s^2 + a^2} ds = 0$$

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From these results and by setting $a = y$

$$\phi(x, y) = \frac{y}{\pi} \cos(x) \int_{-\infty}^{\infty} \frac{\cos(s)}{s^2 + y^2} ds - \frac{y}{\pi} \sin(x) \int_{-\infty}^{\infty} \frac{\sin(s)}{s^2 + y^2} ds$$

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$$\downarrow$$

$$\phi(x, y) = \frac{y}{\pi} \cos(x) \frac{\pi e^{-y}}{y} - 0$$

b. Dirichlet Problem

Final solution to the given Dirichlet problem:

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To confirm solution, check that Laplace's Equation is satisfied

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and that the boundary condition is satisfied, $\phi(x,0) = \cos(x)$.

References

Dennis Zill and Patrick Shanahan (2015)
Complex Analysis: A First Course with Applications 3rd Ed.
Some Consequences of the Residue Theorem Section 6.6.
Poisson Integral Formula Section 7.4.

Thank You