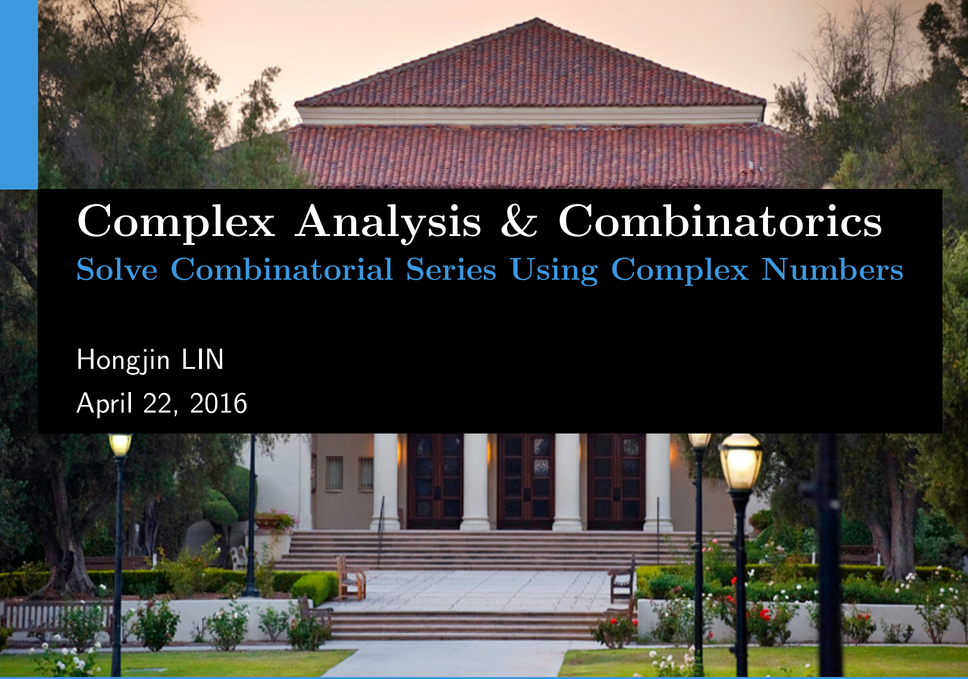

Complex Analysis & Combinatorics

Solve Combinatorial Series Using Complex Numbers

Hongjin LIN

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Outline

1. Goal
2. Definitions
3. Solving the problem using complex numbers
4. Roots of Unity Filter

Goal

Solve the following Combinatorial Series:

$$S_1 = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = \sum_{j=0}^{n} (-1)^j \binom{n}{2j}$$

$$S_2 = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \ldots = \sum_{j=0}^{n} (-1)^j \binom{n}{2j+1}$$

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$\binom{999}{333} = \frac{999!}{333!(999-333)!} = \frac{999!}{333!(666)!} =$

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Result:

38 526 817 469 494 836 637 415 833 778 075 843 488 004 184 990 066 410 152 589 \
969 547 371 490 221 450 419 753 869 304 520 019 987 785 101 443 227 790 478 419 \
793 357 060 474 799 644 437 143 987 497 376 630 741 744 688 541 509 589 605 216 \
946 055 975 393 048 807 187 197 070 705 706 437 278 269 803 982 778 648 046 765 \
448 260 924 108 662 559 517 540 696 615 654 644

$$\binom{999}{333} = \frac{999!}{333!(999-333)!} = \frac{999!}{333!(666)!} =$$

Result:

38 526 817 469 494 836 637 415 833 778 075 843 488 004 184 990 066 410 152 589 ⋱
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Complex Numbers come to your rescue!

Definitions

- **Combinations**: A way of selecting several things out of a larger group, where order does not matter.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

(repetition not allowed)

- **Exponential Form of Complex Numbers**:

$$z = |z|e^{i\theta}, where\ z \in \mathbb{C}\ and\ \theta = arg(z).$$

- **Euler's Formula**:

$$e^{i\theta} = cos(\theta) + isin(\theta)$$

Solving the problem using complex numbers

- Consider:

$$(1-i)^n = \sum_{j=0}^{n} \binom{n}{j} (-i)^j (1)^{n-j} = \sum_{j=0}^{n} \binom{n}{j} (-i)^j$$

(derived from the **Binomial Theorem**)

- Note:

$$(-i)^0 = 1,\ (-i)^1 = -i,\ (-i)^2 = -1,$$
$$(-i)^3 = i,\ (-i)^4 = 1,\ (-i)^5 = -i,\ ...$$
$$\Downarrow$$
$$(-i)^{n+4} = (-i)^n$$

- Expand:

$(1-i)^n$

$$= \sum_{j=0}^{n} \binom{n}{j} (-i)^j$$

$$= \binom{n}{0} - i\binom{n}{1} - \binom{n}{2} + i\binom{n}{3} + \binom{n}{4} - i\binom{n}{5} - \binom{n}{6} + i\binom{n}{7} + \ldots$$

$$= \{\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots\} - i\{\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \ldots\}$$

$$= S_1 - iS_2$$

- Therefore:

$$S_1 = \Re((1-i)^n)$$

$$S_2 = -\Im((1-i)^n)$$

- From the **exponential form** of complex numbers and the **Euler's Formula**, we know:

$$1 - i = \sqrt{2}e^{-\frac{\pi}{4}i}$$

$$(1-i)^n = (\sqrt{2})^n e^{-\frac{n\pi}{4}i}$$

$$= (\sqrt{2})^n cos(-\frac{n\pi}{4}) + i(\sqrt{2})^n sin(-\frac{n\pi}{4})$$

- Therefore:

$$S_1 = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \dots$$
$$= \Re((1-i)^n)$$
$$= \boxed{(\sqrt{2})^n cos(-\frac{n\pi}{4})}$$

$$S_2 = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \binom{n}{7} + \dots$$
$$= -\Im((1-i)^n)$$
$$= \boxed{-(\sqrt{2})^n sin(-\frac{n\pi}{4})}$$

- Check:
  $n = 3$:

$$S_1 = \binom{3}{0} - \binom{3}{2} = 1 - 3 = -2$$

$$(\sqrt{2})^3 cos(-\frac{3\pi}{4}) = 2\sqrt{2}(-\frac{\sqrt{2}}{2}) = -2\checkmark$$

$$S_2 = \binom{3}{1} - \binom{3}{3} = 3 - 1 = 2$$

$$-(\sqrt{2})^3 sin(-\frac{3\pi}{4}) = -(2\sqrt{2})(-\frac{\sqrt{2}}{2}) = 2\checkmark$$

□

Roots of Unity Filter

Definition

Root of Unity

An n^{th} root of unity, where n is a positive integer (i.e. n = 1,2,3,), is a number z satisfying the equation:

$$z^n = 1$$

Since $1 = e^{2\pi i}$, we can write an n^{th} root as $w = e^{2\pi i/n}$. All other n^{th} roots are given by the integer powers of this root between 0 and $n-1$: the roots are 1,w,w^2,...,w^{n-1}. By factoring $z^n - 1 = 0$, we can obtain a useful identity:

$$z^{n1} = (z-1)(z^{n-1} + z^{n-2} + ... + z + 1) = 0$$

This means that all n^{th} roots of unity other than 1 satisfy the equation:

$$1 + w + w^2 + ... + w^{n-1} = 0$$

- **Roots of Unity Filter**:
  Roots of Unity Filter is useful when we are trying to find a specific set of coefficients of a polynomial function.

  Let $f(x)$ be a polynomial function:

  $$f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$$

  - the sum of coefficients:

    $$a_n + a_{n-1} + ... + a_2 + a_1 + a_0 = f(1)$$

    .
  - the sum of coefficients whose terms have even exponents:

    $$\frac{f(1) + f(-1)}{2}$$

  - what about the sum of coefficients whose exponents are multiples of three?

Solving Problem Using Roots of Unity Filter

- *Find the sum of the coefficients of the polynomial* $f(x) = (x+1)^{99}$ *for all powers of x that are divisible by 3.*

  This question is essentially asking us to solve the combinatorial equation (from the **Binomial Theorem**):

  $$\sum_{j=0}^{33} \binom{99}{3j},$$

  since $(x+1)^{99} = 1 + \binom{99}{1}x + \binom{99}{2}x^2 ... + \binom{99}{99}x^{99} = \sum_{j=0}^{99} \binom{99}{j}x^j$