# Complex Hermitian Matrices

Jeh Johnson

Occidental College '17

April 27, 2016

# Overview

- Types of complex matrices
- Properties of the *Hermitian* matrix
- Ways in the Hermitian matrix can be useful

# Complex Matrices

- Hermetian Matrix $A = \overline{A^T}$
- Skew-Hermitian Matrix $-A = \overline{A^T}$
- Unitary Matrix $A^{-1} = \overline{A^T}$

# The Hermitian Matrix

**Definition**: A Hermitian Matrix is a square matrix with complex entries that is equal to its own complex conjugate. We write this as: $A = \overline{A^T}$

**Properties of a Hermitian Matrix**

- The values on the diagonal are always real.
- The matrix cannot be symmetric if it has complex values.
- The matrix must be symmetric if it has only real values.
- All Eigenvalues of a Hermitian matrix are real.
- The sum of two Hermitian matrices is a Hermitian matrix.
- The product of two Hermitian matrices is a Hermitian matrix iff $AB = BA$. Meaning $A^n = Hermitian$.
- The inverse of an invertible Hermitian matrix is Hermitian.

# Examples of Hermitian Matrices

Let

$$A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3+2i & 1 \\ 3-2i & 4 & 2i \\ 1 & -2i & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

Note: All of the entries on the diagonal are real as they must equal the complex conjugate of themselves. And the matrices with complex entries are not symmetric.

# Examples of Hermitian Matrices

$A = \overline{A^T}$

$$A = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 1+i & 2 \\ 1-i & 3 & -i \\ 2 & i & 0 \end{bmatrix}$$

$$\overline{A^T} = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$$

The Sum of Hermitian matrices is Hermitian:

$$A + B = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3+2i & 1 \\ 3-2i & 4 & 2i \\ 1 & -2i & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} (1+2) & (1-i)+(3+2i) & (2+1) \\ (1+i)+(3-2i) & (3+4) & (i+3i) \\ (2+1) & (-i-2i) & (0+1) \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 4+i & 3 \\ 4-i & 7 & 3i \\ 3 & -3i & 1 \end{bmatrix}$$

A+B by defintion is a Hermitian matrix. Notice the values on the diagonal as well as each element equals the complex conjugate of its transpose.

Product of Hermitian matrices is Hermitian $\Leftrightarrow$ AB=BA:
let $n = 2$

$$A^n = \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix} \begin{bmatrix} 1 & 1-i & 2 \\ 1+i & 3 & i \\ 2 & -i & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 4-6i & 3+i \\ 4+6i & 12 & 2+5i \\ 3-i & 2-5i & 5 \end{bmatrix}$$

$A^2$ by definition is a Hermitian matrix. Notice the values on the diagonal as well as each element equals the complex conjugate of its transpose.

# How are Hermitian Matrices Useful?

By identifying upon visual inspection if the matrix has complex values...

- with real values along the diagonal
- with the element $i^{th}$ column and the $j^{th}$ row equaling the complex conjugate of the element in the $j^{th}$ column and the $i^{th}$ row

Then we can assume the further properties of a Hermitian matrix. Such that the matrix will always have real eigenvalues, the sum of Hermitian matrices will also be Hermitian, the product of Hermitian matrices will be Hermitian iff AB=BA, and the inverse of an invertible Hermitian matrix will be Hermitian.

Muchas Gracias