Complex Matrices

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Different Types of Complex Matrices

1. A *Hermitian Matrix* is a matrix where $\bar{A}^T = A$.
2. A *Skew-Hermitian Matrix* is a matrix where $\bar{A}^T = -A$.
3. A *Unitary Matrix* is a matrix where $\bar{A}^T = A^{-1}$.

To find the **complex conjugate transpose**:

- Take the **complex conjugate** of the entries of a $n \times n$ matrix. If the entry is $a + ib$, then the complex conjugate of that entry is $a - ib$
- Find the **transpose** of the matrix, which is obtained by exchanging rows and columns of the matrix

If the matrix's complex conjugate transpose is equal to the inverse, then the matrix is unitary!

Unitary Matrix: Example

Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2+i}{\sqrt{10}} & \frac{-2+i}{\sqrt{10}} \\ 0 & \frac{2+i}{\sqrt{10}} & \frac{2-i}{\sqrt{10}} \end{pmatrix}$$

To determine whether or not this matrix is unitary, we must find the complex conjugate transpose and the inverse.

$$\overline{A}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2-i}{\sqrt{10}} & \frac{2-i}{\sqrt{10}} \\ 0 & \frac{-2-i}{\sqrt{10}} & \frac{2+i}{\sqrt{10}} \end{pmatrix}$$

Unitary Matrix: Example

The matrix is 3×3, so we can find the inverse by finding the determinant and then multiplying by the determinants of each **minor matrix**, which are obtained by eliminating one row and one column and taking the determinant of the resulting 2×2 matrix.

$$det(A) = 1\left(\frac{2+i}{\sqrt{10}}\right)\left(\frac{2-i}{\sqrt{10}}\right) - \left(\frac{-2+i}{\sqrt{10}}\right)\left(\frac{2+i}{\sqrt{10}}\right) = \frac{4+4+1+1}{10} = 1$$

Then, the inverse is represented as

$$A^{-1} = \frac{1}{1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2-i}{\sqrt{10}} & \frac{2-i}{\sqrt{10}} \\ 0 & \frac{-2-i}{\sqrt{10}} & \frac{2+i}{\sqrt{10}} \end{pmatrix}$$

This is equivalent to the Complex Conjugate Transpose in the previous slide, so the matrix A is unitary!

Proof About Unitary Matrices

We will prove that the modulus of a unitary matrix is equal to 1, by showing that $|det(A)| = 1$.

Proof: We will assume that the eigenvalues of a unitary matrix are **unimodular** ($|\lambda| = 1$).Then, say matrix A has n distinct eigenvalues. Then, $|det(A)| = |\lambda_1||\lambda_2| \cdots |\lambda_n|$. Because the products of the eigenvalues of a matrix are equal to the determinant,

$$|det(A)| = |\lambda_1||\lambda_2| \cdots |\lambda_n| = 1 \times 1 \times \cdots \times 1 = 1$$

Therefore, the determinant of a unitary matrix has a modulus equal to 1.