A Case of Complex Matrices and Their Always Real Eigenvalues

Abby Appel and Rebecca Reese

April 22nd, 2016

In our project we draw on comparisons between Linear Algebra and Complex Analysis, by investigating matrices with complex entries and real eigenvalues. In doing this, we will answer questions previously left unanswered in Linear Algebra coursework, by finding and proving specific matrix parameters that always produce real eigenvalues.

We began investigating under which parameters a $2X2$ complex matrix resulted in real eigenvalues. While researching this topic, we came up with some examples of matrices with all complex entries that resulted in real eigenvalues but we couldn't find a pattern as to what format of complex entries *always* produced real eigenvalues. We then came across Hermitian Matrices, which have concrete properties that ensure real eigenvalues from a complex matrix. We therefore decided to investigate the special case of Hermitian Matrices and how they pertain to eigenvalues.

Definition: A *Hermitian Matrix* is a square matrix with complex entries that is equal to its own conjugate transpose.

Definition: A *conjugate transpose* ($(A^*)_{ij}$) of a matrix, also known as a *Hermitian transpose* (A^H), is obtained by taking the transpose of the matrix and then taking the complex conjugate of each entry: ($\overline{A_{ji}}$).

If we look at a matrix with entirely complex entries A=$\begin{bmatrix} 1+3i & 1+i \\ 4-i & 5+2i \end{bmatrix}$, we can determine if it is a Hermitian Matrix by using the formula $A = A^*$
(A=A conjugate transpose).
We first take the transpose of matrix A:

$$A^T = \begin{bmatrix} 1+3i & 4-i \\ 1+i & 5+2i \end{bmatrix}$$

We then take the complex conjugate of each entry of the transpose to obtain the conjugate transpose:

$$A^* = \begin{bmatrix} 1-3i & 4+i \\ 1-i & 5-2i \end{bmatrix}$$

Because A=$\begin{bmatrix} 1+3i & 1+i \\ 4-i & 5+2i \end{bmatrix} \neq \begin{bmatrix} 1-3i & 4+i \\ 1-i & 5-2i \end{bmatrix} = A^*$, matrix A is not Hermitian. We can see rather simply, however, that in order for this equation to ever be true, the diagonal of matrix A must have real entries, otherwise the complex conjugate will never be equal to the original matrix.

Now let's look at matrix B where the diagonal entries are real: $\begin{bmatrix} 1 & 1+i \\ 4-i & 2 \end{bmatrix}$

Let's take the transpose of matrix B:

$$B^T = \begin{bmatrix} 1 & 4-i \\ 1+i & 2 \end{bmatrix}$$

And the complex conjugate of each entry:

$$B^* = \begin{bmatrix} 1 & 4+i \\ 1-i & 2 \end{bmatrix}$$

Because B=$\begin{bmatrix} 1 & 1+i \\ 4-i & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 4+i \\ 1-i & 2 \end{bmatrix} = B^*$, matrix B is also not Hermitian due to the complex entries inequality. If, however, the diagonal has purely real entries and the complex entries are conjugates of eachother, the transpose conjugate will be equal to the original matrix.

Example: C=$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$

$$C^T = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

$$C^* = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$$

$$C = C^* \checkmark$$

So what makes a matrix with Hermitian format special or interesting? Their eigenvalues. Here, we find the eigenvalues of the Hermitian Matrix C from above.

C=$\begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1-\lambda & 1+i \\ 1-i & 2-\lambda \end{bmatrix} \rightarrow (1-\lambda)(2-\lambda)-(1+i)(1-i)$

$= (2-\lambda-2\lambda+\lambda^2)-(1-i+i+1) = (\lambda^2-3\lambda+2)-(2) = \lambda^2-3\lambda = 0$

$$\lambda(\lambda-3) = 0 \rightarrow \lambda = 0, 3$$

An important fact that comes from Hermitian Matrixes is that though half of its entries can be complex, the eigenvalues that result will *always* be real.

Proof: Recall that the scalar 1 is an eigenvalue of the (square) matrix A if and only if there is a column vector X such that

$$AX = \lambda \mathrm{X}$$

Taking the Hermitian conjugate of both sides, applying identity (2), and noting that multiplication by a scalar is commutative, we have

$$X^H A^H = \lambda^* X^H$$

Now, if A is Hermitian, we have (by definition) $A^H = A$, so this becomes

$$X^H A = \lambda^* X^H$$

If we multiply X by both sides of this equation, and if we multiply both sides of the original eigenvalue equation by XH, we get

$$X^H AX = \lambda^* X^H X \qquad\qquad X^H AX = X^H \lambda X$$

Since the left hand sides are equal, and since multiplication by scalar is commutative, we have 1=1*, and therefore the imaginary part of 1 is 0 making it purely real.
Now we have proven that there is a standard complex matrix form that will always give us real eigenvalues!

Another interesting property of Hermitian matrices is that if you take any complex matrix, including ones with complex diagonals, and you add it to its conjugate transpose, the sum is Hermitian.
Example: $\begin{bmatrix} 3-i & 2-i \\ 1+i & 1+2i \end{bmatrix} + \begin{bmatrix} 3+i & 1-i \\ 2+i & 1-2i \end{bmatrix} = \begin{bmatrix} 6 & 3-2i \\ 3+2i & 2 \end{bmatrix}$=Hermitian ✓

This is really exciting because we now know we have a format of standard matrices that always gives us real eigenvalues. This has an important application to modern physics.