1. "If \( f(\overline{z}) \) is a complex function, then \( f(x+oi) \) must be a real number." **FALSE**

   Clearly, \( f(\overline{z}) = iz \) is an example of a function where \( f(x+oi) = ix \) which is **NOT** a real number.

2. "\( \arg(z) \) is a complex function." **FALSE**

   \( \arg(z) \) outputs multiple values for each input so it is **NOT** a function.

3. "The domain of the function \( f(z) = \frac{1}{z^2+i} \) is all complex numbers." **FALSE**

   The values \( z^2+i = 0 \) are **NOT** in the domain of \( f(z) \).

4. "The domain of the function \( f(z) = e^{z^2} - (1+i)z+2 \) is all complex numbers." **TRUE**

5. "If \( f(\overline{z}) \) is a complex function with \( u(x,y) = 0 \), then the range of \( f \) lies in the imaginary axis." **TRUE**

6. "The entire complex plane is mapped onto the real axis \( v=0 \) by \( w = z + \overline{z} \)." **TRUE**

   \[ w = (x+iy) + (x-iy) = 2x \text{ where } x \in \mathbb{R} \]

   So \( w \in \mathbb{R} \)

7. "The entire complex plane is mapped onto the unit circle \( |w| = 1 \) by \( w = \frac{z}{1+|z|^2} \)." **FALSE**

   \[ |w| = \left| \frac{z}{1+|z|^2} \right| < 1 \]

   \[ |z| = \left| \frac{1}{1+|z|^2} \right| < 1 \]
8. "The range of the function \( f(x) = \text{Arg}(x) \) is all real numbers." \[ \text{FALSE} \]

The range of \( \text{Arg}(x) \) is \( \{ x \in \mathbb{R} : -\pi < x \leq \pi \} \).

9. "The image of the circle \( |z - z_0| = p \) under a linear mapping is a circle with a possibly different center, but the same radius." \[ \text{FALSE} \]

\( w = 2z \) is a linear mapping and it changes \( |z - z_0| = p \) to be \( |w - 2z_0| = 2p \). Center moved and radius increased.

\[ z_{\{1} = z_0 + p e^{i \theta}, \quad 0 \leq \theta \leq 2\pi \]
\[ w = 2z = 2z_0 + 2p e^{i \theta}, \quad 0 \leq \theta \leq 2\pi \]

10. "The linear mapping \( w = (1 - \sqrt{3}i)z^2 + 2 \) is by rotating through an angle of \( \frac{\pi}{3} \) radians clockwise about the origin, magnifying by a factor of 2, then translating by 2. \[ \text{TRUE} \]

\[ R(z) = e^{-\frac{\pi i}{3} z} \]
\[ S(z) = 2z \]
\[ T(z) = 2 \]
\[ w(z) = T(S(R(z))) = T(S(e^{-\frac{\pi i}{3} z})) \]
\[ = T(2e^{-\frac{\pi i}{3} z}) \]
\[ = 2e^{-\frac{\pi i}{3} z} + 2 \]
\[ = 2 \left( \frac{1 - i\sqrt{3}}{2} \right) z + 2 \]
\[ = (1 - i\sqrt{3})z + 2 \]
HW Set 4

3.1.1. 2, 11, 17, 20*

2) \( \lim_{z \to 1+i} \frac{z - \bar{z}}{z + \bar{z}} = \lim_{(x,y) \to (1,1)} u(x,y) + iv(x,y) = 1 \)

\[ \frac{z - \bar{z}}{z + \bar{z}} = \frac{(x+iy) - (x-iy)}{(x+iy) + (x-iy)} = \frac{2iy}{2x} = \frac{iy}{x} \Rightarrow \frac{v}{x} = \frac{y}{x} \]

11) \( \lim_{z \to e^{\frac{3\pi}{4}}} (z + \frac{1}{z}) = e^{\frac{3\pi}{4}} + \frac{1}{e^{\frac{3\pi}{4}}} = e^{\frac{3\pi}{4}} + e^{-\frac{3\pi}{4}} = 2 \cos \left( \frac{3\pi}{4} \right) = \sqrt{2} \)

17) \( \lim_{z \to 0} \frac{\Re (z)}{\Im (z)} = \lim_{(x,y) \to (0,0)} \frac{x}{y} \frac{4}{5} \)

(a) If \( z \to 0 \) along \( y = x \), \( \lim_{z \to 0} \frac{\Re (z)}{\Im (z)} = \lim_{(x,y) \to (0,0)} 1 = 1 \)

(b) If \( z \to 0 \) along imaginary axis \( y = 0 \), \( \lim_{z \to 0} \frac{\Re (z)}{\Im (z)} = \lim_{(x,y) \to (0,0)} 0 = 0 \)

(c) The limit does not exist!

20) \( \lim_{z \to 0} \left( \frac{2y^2 - x^2 - y^2 i}{x^2 - y^2 i} \right) \)

(a) Suppose \( y = x \)
\[ \lim_{(x,y) \to (0,0)} 2 - 0i = 2 \]

(b) Suppose \( y = -x \)
\[ \lim_{(x,y) \to (0,0)} 2 - 0i = 2 \]

(c) Answers imply that limit exists since we get same answer along both paths

(d) Suppose \( y = 2x \)
\[ \lim_{(x,y) \to (0,0)} 2(2x^2 - x^2 - 2xi) \]
\[ = \lim_{(x,y) \to (0,0)} -3i = 8 + 3i \frac{1}{4} \]

(e) Limit does not exist!
28. \( f(z) = \frac{z^3 - \frac{1}{z}}{z} \quad z_0 = 3i \)

\[
\lim_{z \to 3i} f(z) = (3i)^3 - \frac{1}{3i} = -27i - \frac{1}{3}(-i) = -80i
\]

31. \( f(z) = \begin{cases} 
\frac{z^3}{z-1}, & |z| \neq 1 \\
\frac{z^3}{3}, & |z| = 1
\end{cases} \)

\[
\lim_{z \to 1} \frac{(z-1)(z^2+1)}{(z-1)} = \lim_{z \to 1} z^2 + 1 = 2
\]

37. \( \lim_{z \to -1} \arg z = \text{DNE} \)

\[
\lim_{z \to -1} \arg z = \pi \quad \text{when } z \to -1 \text{ from above}
\]

\[
\lim_{z \to -1} \arg z = -\pi \quad \text{when } z \to -1 \text{ from below}
\]

50. \( \lim_{z \to z_0} \bar{z} = \bar{z}_0 \) if for every \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( |\bar{z} - \bar{z}_0| < \epsilon \) whenever \( 0 < |z - z_0| < \delta \).

By properties of complex modulus and conjugation, \( |z - z_0| = |\bar{z} - \bar{z}_0| = |\bar{z} - \bar{z}_0| \). Therefore, if \( 0 < |z - z_0| < \delta \) and \( \delta = \epsilon \), then \( |\bar{z} - \bar{z}_0| < \epsilon \).