## EXERCISES 3.1

Answers to selected odd-numbered problems begin on page ANS-10.

### 3.1.1 Limits

In Problems 1–8, use Theorem 3.1.1 and the properties of real limits on page 103 to compute the given complex limit.

1. $\lim_{z\to 2i}(z^2 - \bar{z})$

2. $\lim_{z\to 1+i} \dfrac{z-\bar{z}}{z+\bar{z}}$

3. $\lim_{z\to 1-i}(|z|^2 - i\bar{z})$

4. $\lim_{z\to 3i} \dfrac{\operatorname{Im}(z^2)}{z+\operatorname{Re}(z)}$

5. $\lim_{z\to \pi i} e^z$

6. $\lim_{z\to 0} \dfrac{z^2+\bar{z}^2}{\operatorname{Re}(z)+\operatorname{Im}(z)}$

7. $\lim_{z\to 0} \dfrac{e^z - e^{\bar{z}}}{\operatorname{Im}(z)}$

8. $\lim_{z\to 1+i}\left(\log_e|x^2+y^2| + i\arctan\dfrac{y}{x}\right)$

In Problems 9–16, use Theorem 3.1.2 and the basic limits (15) and (16) to compute the given complex limit.

9. $\lim_{z\to 2-i}(z^2 - z)$

10. $\lim_{z\to i}(z^5 - z^2 + z)$

11. $\lim_{z\to e^{i\pi/4}}\left(z+\dfrac{1}{z}\right)$

12. $\lim_{z\to 1+i} \dfrac{z^2+1}{z^2-1}$

13. $\lim_{z\to -i} \dfrac{z^4-1}{z+i}$

14. $\lim_{z\to 2+i} \dfrac{z^2-(2+i)^2}{z-(2+i)}$

15. $\lim_{z\to z_0} \dfrac{(az+b)-(az_0+b)}{z-z_0}$

16. $\lim_{z\to -3+i\sqrt{2}} \dfrac{z+3-i\sqrt{2}}{z^2+6z+11}$

17. Consider the limit $\lim_{z\to 0} \dfrac{\operatorname{Re}(z)}{\operatorname{Im}(z)}$.
   (a) What value does the limit approach as $z$ approaches 0 along the line $y = x$?
   (b) What value does the limit approach as $z$ approaches 0 along the imaginary axis?
   (c) Based on your answers for (a) and (b), what can you say about $\lim_{z\to 0} \dfrac{\operatorname{Re}(z)}{\operatorname{Im}(z)}$?

18. Consider the limit $\lim_{z\to i}(|z| + i\operatorname{Arg}(iz))$.
   (a) What value does the limit approach as $z$ approaches $i$ along the unit circle $|z| = 1$ in the first quadrant?
   (b) What value does the limit approach as $z$ approaches $i$ along the unit circle $|z| = 1$ in the second quadrant?
   (c) Based on your answers for (a) and (b), what can you say about $\lim_{z\to i}(|z| + i\operatorname{Arg}(iz))$?

19. Consider the limit $\lim_{z\to 0}\left(\dfrac{z}{\bar{z}}\right)^2$.
   (a) What value does the limit approach as $z$ approaches 0 along the real axis?
   (b) What value does the limit approach as $z$ approaches 0 along the imaginary axis?
   (c) Do the answers from (a) and (b) imply that $\lim_{z\to 0}\left(\dfrac{z}{\bar{z}}\right)^2$ exists? Explain.
   (d) What value does the limit approach as $z$ approaches 0 along the line $y = x$?
   (e) What can you say about $\lim_{z\to 0}\left(\dfrac{z}{\bar{z}}\right)^2$?

**20.** Consider the limit $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$.

(a) What value does the limit approach as $z$ approaches 0 along the line $y = x$?

(b) What value does the limit approach as $z$ approaches 0 along the line $y = -x$?

(c) Do the answers from (a) and (b) imply that $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$ exists? Explain.

(d) What value does the limit approach as $z$ approaches 0 along the line $y = 2x$?

(e) What can you say about $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$?

Problems 21–26 involve concepts of infinite limits and limits at infinity discussed in (*i*) of the *Remarks*. In Problems 21–26, use (21) or (22), Theorem 3.1.2, and the basic limits (15) and (16) to compute the given complex limit.

**21.** $\lim_{z\to\infty} \frac{z^2+iz-2}{(1+2i)z^2}$

**22.** $\lim_{z\to\infty} \frac{iz+1}{2z-i}$

**23.** $\lim_{z\to i} \frac{z^2-1}{z^2+1}$

**24.** $\lim_{z\to -i/2} \frac{(1-i)z+i}{2z+i}$

**25.** $\lim_{z\to\infty} \frac{z^2-(2+3i)z+1}{iz-3}$

**26.** $\lim_{z\to i} \frac{z^2+1}{z^2+z+1-i}$

### 3.1.2 Continuity

In Problems 27–34, show that the function $f$ is continuous at the given point.

**27.** $f(z) = z^2 - iz + 3 - 2i;\ z_0 = 2 - i$

**28.** $f(z) = z^3 - \frac{1}{z};\ z_0 = 3i$

**29.** $f(z) = \frac{z^3}{z^3+3z^2+z};\ z_0 = i$

**30.** $f(z) = \frac{z-3i}{z^2+2z-1};\ z_0 = 1+i$

**31.** $f(z) = \begin{cases} \frac{z^3-1}{z-1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases};\ z_0 = 1$

**32.** $f(z) = \begin{cases} \frac{z^3-1}{z^2+z+1}, & |z| \neq 1 \\ \frac{-1+i\sqrt{3}}{2}, & |z| = 1 \end{cases};\ z_0 = \frac{-1+i\sqrt{3}}{2}$

**33.** $f(z) = \bar{z} - 3\,\mathrm{Re}(z) + i;\ z_0 = 3 - 2i$

**34.** $f(z) = \frac{\mathrm{Re}(z)}{z+iz} - 2z^2;\ z_0 = e^{i\pi/4}$

In Problems 35–40, show that the function $f$ is discontinuous at the given point.

**35.** $f(z) = \frac{z^2+1}{z+i};\ z_0 = -i$

**36.** $f(z) = \frac{1}{|z|-1};\ z_0 = i$

**37.** $f(z) = \mathrm{Arg}(z);\ z = -1$

**38.** $f(z) = \mathrm{Arg}(iz);\ z_0 = i$

**39.** $f(z) = \begin{cases} \dfrac{z^3 - 1}{z - 1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases}$ ; $z_0 = i$

**40.** $f(z) = \begin{cases} \dfrac{z}{|z|}, & z \neq 0 \\ 1, & z = 0 \end{cases}$ ; $z_0 = 0$

In Problems 41–44, use Theorem 3.1.3 to determine the largest region in the complex plane on which the function $f$ is continuous.

**41.** $f(z) = \text{Re}(z)\,\text{Im}(z)$

**42.** $f(z) = \bar{z}$

**43.** $f(z) = \dfrac{z - 1}{z\bar{z} - 4}$

**44.** $f(z) = \dfrac{z^2}{(|z| - 1)\,\text{Im}(z)}$

### Focus on Concepts

**45.** Use Theorem 3.1.1 to prove:

(a) $\lim_{z \to z_0} c = c$, where $c$ is a constant.

(b) $\lim_{z \to z_0} z = z_0$.

**46.** Use Theorem 3.1.1 to show that $\lim_{z \to z_0} \bar{z} = \bar{z}_0$.

**47.** Use Theorem 3.1.2 and Problem 46 to show that

(a) $\lim_{z \to z_0} \text{Re}(z) = \text{Re}(z_0)$.

(b) $\lim_{z \to z_0} \text{Im}(z) = \text{Im}(z_0)$.

(c) $\lim_{z \to z_0} |z| = |z_0|$.

**48.** Use Theorem 3.1.1 to prove part *(ii)* of Theorem 3.1.2.

**49.** The following is an epsilon-delta proof that $\lim_{z \to z_0} z = z_0$. Fill in the missing parts.

**Proof** By Definition 3.1.1, $\lim_{z \to z_0} z = z_0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\underline{\quad\quad}| < \varepsilon$ whenever $0 < |\underline{\quad\quad}| < \delta$. Setting $\delta =$ ______ will ensure that the previous statement is true.

**50.** The following is an epsilon-delta proof that $\lim_{z \to z_0} \bar{z} = \bar{z}_0$. Provide the missing justifications in the proof.

**Proof** By Definition 3.1.1, $\lim_{z \to z_0} \bar{z} = \bar{z}_0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\underline{\quad\quad}| < \varepsilon$ whenever $0 < |\underline{\quad\quad}| < \delta$. By properties of complex modulus and conjugation, $|z - z_0| = |\overline{z - z_0}| = |\underline{\quad\quad}|$. Therefore, if $0 < |z - z_0| < \delta$ and $\delta =$ ______, then $|\bar{z} - \bar{z}_0| < \varepsilon$.

**51.** In this problem we will develop an epsilon-delta proof that $\lim_{z \to 1+i} ((1 - i)z + 2i) = 2 + 2i$.

(a) Write down the epsilon-delta definition (Definition 3.1.1) of $\lim_{z \to 1+i} [(1 - i)z + 2i] = 2 + 2i$.

(b) Factor out $(1 - i)$ from the inequality involving $\varepsilon$ (from part (a)) and simplify. Now rewrite this inequality in the form $|z - (1 + i)| <$ ______.

(c) Based on your work from part (b), what should $\delta$ be set equal to?

(d) Write an epsilon-delta proof that $\lim_{z \to 1+i} [(1 - i)z + 2i] = 2 + 2i$.

**52.** In this problem we will develop an epsilon-delta proof that $\lim_{z \to 2i} \dfrac{2z^2 - 3iz + 2}{z - 2i} = 5i$.

(a) Write down the epsilon-delta definition (Definition 3.1.1) of $\lim_{z \to 2i} \dfrac{2z^2 - 3iz + 2}{z - 2i} = 5i$.

(b) Simplify the inequality involving $\varepsilon$ (from part (a)), then rewrite this inequality in the form $|z - 2i| <$ ______.

## 3.2 Differe