HW 12

6.2 3, 15, 20, 24
6.3 7, 8, 9, 10

6.2.3 $f(z) = \dfrac{1}{(1+2z)^2}$

$$\frac{1}{1+z} \approx \sum_{k=0}^{\infty} (-1)^k z^k \approx 1 - z + z^2 - z^3 + z^4 - z^5 + \ldots$$

$$\frac{-1}{(1+z)^2} \approx \sum_{k=1}^{\infty} (-1)^k k z^{k-1} \approx -1 + 2z - 3z^2 + 4z^3 - 5z^4 + \ldots$$

$$\frac{1}{(1+2z)^2} \approx \sum_{k=1}^{\infty} (-1)^{k+1} k (2z)^{k-1} \approx 1 + 4z - 12z^2 + 32z^3 - 80z^4 + \ldots$$

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+2}(k+1)(2z)^k}{(-1)^{k+1} k (2z)^{k-1}} \right| = \lim_{k\to\infty} \frac{k+1}{k} |2z| = |2z| < 1 \Rightarrow |z| < \frac{1}{2}$$

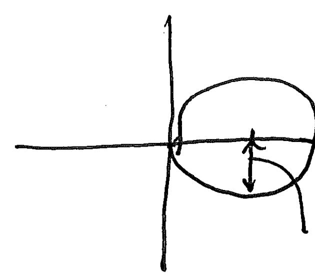
$R = 1/2$

6.2.15
$f(z) = \frac{1}{z}$, $z_0 = 1$ $f'(z) = -\frac{1}{z^2}$ $f''(z) = \frac{2}{z^3}$ $f'''(z) = -\frac{6}{z^4}$

$$f^{(k)}(z) = \frac{k!(-1)^k}{z^{k+1}} \qquad f^{(k)}(1) = k!(-1)^k$$

$$f(z) = \sum_{k=0}^{\infty} f^{(k)}(1) \frac{(z-1)^k}{k!} = \sum_{k=0}^{\infty} k!(-1)^k \frac{(z-1)^k}{k!} = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}(z-1)^{k+1}}{(-1)^k (z-1)^k} \right| = \lim_{k\to\infty} |z-1| = |z-1| < 1$$



R=1

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6.2.20

$f(z) = \frac{1+z}{1-z}$ about $z_0 = i$

$f'(z) = \frac{(1-z)\cdot 1 - (1+z)(-1)}{(1-z)^2} = \frac{1-z+1+z}{(1-z)^2} = \frac{2}{(1-z)^2}$

$f''(z) = \frac{2\cdot -2}{(1-z)^3}(-1) = \frac{2^2}{(1-z)^3}$ $\quad f'''(z) = \frac{4\cdot(-3)}{(1-z)^4}(-1) = \frac{12}{(1-z)^4} = \frac{2\cdot 3!}{(1-z)^4}$

$f^{IV}(z) = \frac{12\cdot(-4)\cdot(-1)}{(1-z)^5} = \frac{48}{(1-z)^5} = \frac{2\cdot 4!}{(1-z)^5}$

$f^{(k)}(z) = \frac{2\cdot k!}{(1-z)^{k+1}}$ $\quad f^{(k)}(i) = \frac{2\cdot k!}{(1-i)^{k+1}}$

$f(i) = \frac{1+i}{1-i} = \frac{(1+i)^2}{1+1} = \frac{2i}{2} = i$ $\quad f'(i) = \frac{2}{(1-i)^2} = \frac{2}{-2i} = i$

$\sum_{k=0}^{\infty} f^{k}(i)\frac{(z-i)^k}{k!} = \sum_{k=0}^{\infty} \frac{2\cdot k!}{(1-i)^{k+1}}\frac{(z-i)^k}{k!} = \sum_{k=0}^{\infty}\frac{2}{(1-i)^{k+1}}(z-i)^k$

$\lim_{k\to\infty}\left|\frac{\frac{2}{(1-i)^{k+2}}(z-i)^{k+1}}{\frac{2}{(1-i)^{k+1}}(z-i)^k}\right| = \lim_{k\to\infty}\frac{|z-i|}{|1-i|} = \frac{|z-i|}{\sqrt{2}} < 1$

$\Rightarrow |z-i| < \sqrt{2}$

So $R = \sqrt{2}$

6.2.24

$f(z) = e^{\frac{1}{1+z}} = 1 + \frac{1}{1+z} + \left(\frac{1}{1+z}\right)^2 + \dots$

$f(0) = e^{\frac{1}{1+0}} = e$

$f'(z) = e^{\frac{1}{1+z}}\cdot\frac{-1}{(1+z)^2}$ $\quad f'(0) = e\cdot\frac{-1}{(1+0)^2} = -e$

$f''(z) = e^{\frac{1}{1+z}}\cdot\frac{1}{(1+z)^2} + e^{\frac{1}{1+z}}\cdot\frac{2}{(1+z)^3}$ $\quad f''(0) = e\cdot\frac{1}{1} + e\cdot\frac{2}{1} = 3e$

$f(z) \approx e - ez + 3e\frac{z^2}{2!} + \dots$

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6.3.7

$f(z) = \frac{1}{z(z-3)}$ Laurent series for $0<|z|<3$

$\Rightarrow 0 < \left|\frac{z}{3}\right| < 1$

$$= -\frac{1}{3z}\frac{1}{(1-\frac{z}{3})} = -\frac{1}{3z}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+\cdots\right]$$

$$= -\frac{1}{3z} - \frac{1}{3^2} - \frac{z^1}{3^3} - \frac{z^2}{3^4} - \cdots = -\sum_{k=0}^{\infty} \frac{z^{k-1}}{3^{k+1}}$$

Laurent Series for $|z|>3 \Rightarrow 1 > \left|\frac{3}{z}\right|$

6.3.9 $\frac{1}{z(z-3)}$

$$= \frac{1}{z^2-3z} = \frac{1}{z^2}\frac{1}{(1-\frac{3}{z})} = \frac{1}{z^2}\left[1+\frac{3}{z}+\left(\frac{3}{z}\right)^2+\left(\frac{3}{z}\right)^3+\cdots\right]$$

$$= \frac{1}{z^2}+\frac{3}{z^3}+\frac{3^2}{z^4}+\frac{3^3}{z^5}+\cdots = \sum_{k=0}^{\infty} 3^k z^{-k-2}$$

6.3.8 $0<|z-3|<3 \Rightarrow 0 < \left|\frac{z-3}{3}\right| < 1$

$$\frac{1}{(3+z-3)}\frac{1}{(z-3)} = \frac{1}{3}\frac{1}{1+\frac{(z-3)}{3}}\cdot\frac{1}{(z-3)} = \frac{1}{3}\cdot\frac{1}{z-3}\left[1-\frac{z-3}{3}+\left(\frac{z-3}{3}\right)^2+\cdots\right]$$

$$= \frac{1}{3}\frac{1}{z-3} - \frac{1}{9} + \frac{1}{3^3}(z-3) - \frac{1}{3^4}(z-3)^2 + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(z-3)^{k-1}}{3^{k+1}}(-1)^k$$

6.3.10 $3 < |z-3| \Rightarrow \frac{3}{|z-3|} < 1$

$$\frac{1}{z}\left(\frac{1}{z-3}\right) = \frac{1}{\left(\frac{3}{z-3}+1\right)}\frac{1}{(z-3)^2} = \frac{1}{(z-3)^2}\left[1-\left(\frac{3}{z-3}\right)+\left(\frac{3}{z-3}\right)^2-\left(\frac{3}{z-3}\right)^3+\cdots\right]$$

$$= \frac{1}{(z-3)^2} - \frac{3}{(z-3)^3} + \frac{3^2}{(z-3)^4} - \frac{3^3}{(z-3)^5} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{3^k}{(z-3)^{k+2}}$$