HW 10
MATH 312

Chap 5 : 4, 5, 6, 7, 8, 9, 17, 38*, 40*
6.4 : 2, 6, 20
6.5 : 2, 12, 17, 23
6.6 : 3, 8, 15, 24*

4. "If f is analytic at z_0, then f''' necessarily exists at z_0."

$\boxed{\text{TRUE}}$ If a function is analytic it is infinitely differentiable at

5. "If f is analytic within and on a simple closed contour C and z_0 is any point within C, then the value of $f(z_0)$ is determined by values of $f(z)$ on C."

$\boxed{\text{TRUE}}$ $f(z_0) = \frac{1}{2\pi i}\oint_C f(z)\,dz$

6 "If f is analytic on a simple closed contour then $\oint_C f(z)\,dz = 0$."

$\boxed{\text{FALSE}}$ f has to be analytic ON <u>AND</u> INSIDE a simple closed contour

7 "If f is continuous on a domain D and has an antiderivative F in D then $\int f(z)\,dz$ has the same value on all contours C in D between z_0 and z_1"

$\boxed{\text{TRUE}}$ Path Invariance Theorem

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8. "If $\oint_C f(z)\,dz = 0$ for every simple closed contour C then f is analytic within and on C."

$\boxed{\text{FALSE}}$ ~~TRUE~~ If you have a function like $\oint_{|z|=2} \frac{1}{z^2+1}\,dz = 0$ but is not analytic inside $|z|=2$

9. "The value of $\int_C z - \frac{2}{z}$ is the same for any path C in the right half plane $\mathrm{Re}\, z > 0$ between $z = 1+i$ and $z = 10+8i$."

$\boxed{\text{TRUE}}$ Path independence theorem.

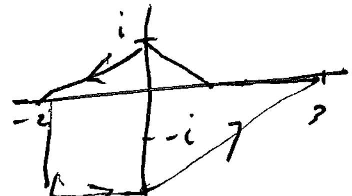
$f(z) = z - \frac{2}{z}$ is analytic in $\mathrm{Re}\, z > 0$

so $\int_{z_1}^{z_2} f(z)\,dz = F(z_2) - F(z_1)$ where $F'(z) = f(z)$

17. $\int_{-i}^{i} \frac{1}{z}\,dz = \mathrm{Ln}(i) - \mathrm{Ln}(-i) = \frac{\pi i}{2} - -\frac{\pi i}{2} = \pi i$ $\boxed{\text{TRUE}}$

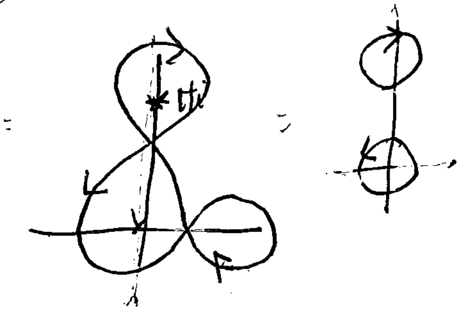
where $-i$ to i is any path that avoids the origin and negative real axis.

38. $\oint_C \frac{z}{z+i}\,dz = \boxed{0}$ C =



40. $\oint_C \frac{e^z}{z^2(z-\pi i)}\,dz = 2\pi i \frac{e^{\pi i}}{(\pi i)^2}$

$= \boxed{\frac{2}{\pi}}$ C =



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Sec 64: (2), (6), (20)

(2) $f(z) = \dfrac{z^3 - 4z^2}{1 - e^{z^2/2}} = \dfrac{z^2(z-4)}{1 - e^{z^2/2}}$

$\lim_{z \to 0} f(z) \overset{LR}{=} \dfrac{3z^2 - 8z}{-ze^{z^2/2}} = \lim_{z \to 0} \dfrac{3z - 8}{-e^{z^2/2}} = 8$

$f(z) = \begin{cases} 8, & z = 0 \\ \dfrac{z^3 - 4z^2}{1 - e^{z^2/2}}, & z \neq 0 \end{cases}$

(6) $f(z) = z^4 - 16 = 0 \Rightarrow z = \sqrt[4]{16} = 2e^{\pi i/2}, 2, 2e^{\pi i}, 2e^{-\frac{\pi i}{2}}$

zeroes of order 1

$f'(z) = 4z^3$

$f'(2) = 32, \ f'(2i) = -32i \neq 0$

$f'(-2) = -32 \neq 0, \ f'(-2i) = 32i \neq 0$

(20) $f(z) = \dfrac{\cot \pi(z)}{z^2} = \dfrac{\cos(\pi z)}{z^2 \sin(z)}$

$z = 0$ is a pole of order 3

$z^2 \sin z$ has a zero of order 3 at $z = 0$

$g = z^2 \sin z \quad g'(z) = 2z \sin z + z^2 \cdot \cos z$

$g(0) = 0 \quad g'(0) = 0$

$g''(z) = 2\sin z + 2z\cos z + 2z\cos z + z^2 \sin z \quad g'''(0) = 6 \neq 0$

$g''(0) = 0 \quad g'''(z) = 2\cos z + 4\cos z + 4z\sin z - 2z\sin z + z^2 \sin z$

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Sec 6.5 : 2, 12, 17, 23

2. $f(z) = \dfrac{1}{z^3(1-z)^3}$

$\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \dots = (1 + z + z^2 + O(z^3))^3$

$\dfrac{1}{(1-z)^3} = 1^3 + z^3 + z^6 + 3z + 3z^2 + \cancel{3z^3}\ 3(1+z+z^2+\dots) + 3(z+z^2)^2 + (z^3+z^2)^3$

$(A+B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

$\dfrac{1}{(1-z)^3} \approx 1 + 3 + 3z + 3z^2 + 3z^3 + 3z^4 + \dots + 3(z^2 + 2z^3 + z^4 + \dots) + z^3 + 3z^3 + \dots$

$\dfrac{1}{(1-z)^3} \approx 4 + 3z + 6z^2 + 12z^3 + \dots$

$\dfrac{1}{z^3}\left(\dfrac{1}{1-z}\right)^3 \approx \dfrac{4}{z^3} + \dfrac{3}{z^2} + \boxed{\dfrac{6}{z}} + 12 + \dots$ coefficient is 6 ← Residue

$\operatorname{Res}\left(\dfrac{1}{z^3(1-z)^3}, 0, 3\right)$ order

$\lim\limits_{z \to 0} \dfrac{1}{2!}\dfrac{d^2}{dz^2}\left(z^3 \cdot \dfrac{1}{z^3(1-z)^3}\right) = \dfrac{1}{2}\lim\limits_{z \to 0} \dfrac{12}{(1-z)^5} = \dfrac{12}{2} = 6$

$f = (1-z)^{-3}$

$f' = -3(1-z)^{-4} \cdot -1$

$f''(z) = +3 \cdot -4 \cdot (1-z)^{-5} \cdot -1$

$f''(z) = 12(1-z)^{-5}$

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6.5 (2), (12) (17), (23)

(12) $f(z) = \dfrac{2z-1}{(z-1)^4(z+3)}$

This function has a pole of order 4 at $z=1$ and a simple pole at $z=-3$

$$\mathrm{Res}(f,-3,1) = \frac{2(-3)-1}{4(-3-1)^3\cdot(-3+3)+(-3-1)^4\cdot 1}$$

$$= \frac{-7}{256}$$

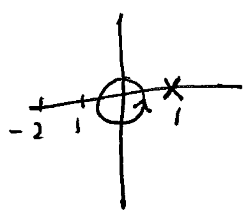
$$\mathrm{Res}(f,1,4) = \lim_{z\to 1}\frac{1}{3!}\frac{d^3}{dz^3}\left[(z-1)^4\frac{1}{(z+3)(z-1)^4}\right]$$

$$= \lim_{z\to 1}\frac{1}{6}\frac{d^2}{dz^2}\frac{-1}{(z+3)^2}$$

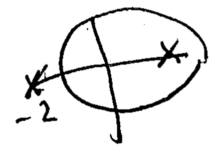
$$= \lim_{z\to 1}\frac{1}{6}\frac{d}{dz}\frac{2}{(z+3)^3} = \lim_{z\to 1}\frac{1}{6}\cdot\frac{-6}{(z+3)^4}$$

$$= \frac{-1}{256}$$

(17) (a) $\oint_C \dfrac{1}{(z-1)(z+2)^2}\,dz = 0$ when $|z| = \frac{1}{2}$



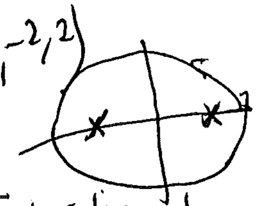
$$= 2\pi i\,\mathrm{Res}\left(\frac{1}{(z-1)(z+2)^2}, 1, 1\right)$$



$$= 2\pi i \lim_{z\to 1}\frac{z-1}{(z-1)(z+2)^2}$$

$\boxed{(b) = 2\pi i\cdot\frac{1}{9}}$

$$= 2\pi i\left(\frac{1}{9} + \mathrm{Res}\left(\frac{1}{(z-1)(z+2)^2}, -2, 2\right)\right)$$



$= 2\pi i\left(\frac{1}{9}-\frac{1}{9}\right)$ $\quad \mathrm{Res} = \lim_{z\to -2}\dfrac{d}{dz}\dfrac{1}{(z-1)} = \lim_{z\to -2}\dfrac{-1}{(z-1)^2}$

$\boxed{(c) = 0}$

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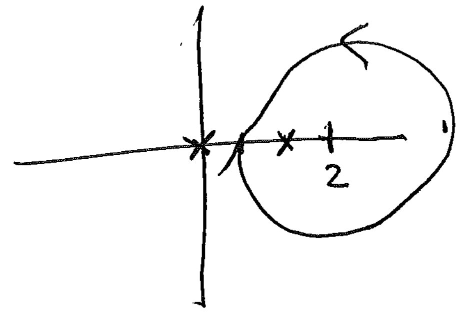
6.5 #22 (EXTRA)

$$\oint_C \frac{1}{z^3}\cdot\frac{1}{(z-1)^4}\,dz$$

$C: |z-2| = \frac{3}{2}$



$$I = 2\pi i \,\mathrm{Res}\left(\frac{1}{z^3(z-1)^4}, 1, 4\right)$$

$$\mathrm{Res}\left(\frac{1}{z^3(z-1)^4}, 1, 4\right) = \lim_{z\to 1}\frac{1}{3!}\frac{d^3}{dz^3}(z-1)^4\frac{1}{z^3}\cdot\frac{1}{(z-1)^4}$$

$$= \lim_{z\to 1}\frac{1}{6}\frac{d^3}{dz^3}\left(z^{-3}\right)$$

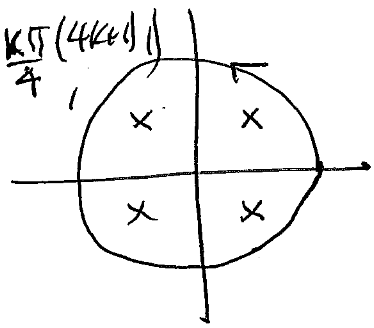
$$= \frac{1}{6}\cdot\left(-3\cdot -4\cdot -5\cdot z^{-6}\right)\Big|_{z=1}$$

$$= -10$$

$$\boxed{I = -20\pi i}$$

6.5.23

$$\oint_{|z|=2}\frac{1}{z^4-1}\,dz = 2\pi i\sum \mathrm{Res}\left(\frac{1}{z^4-1}, e^{\frac{ik\pi}{4}(4k+1)}, 1\right)$$



$= 2\pi i$ (sum of the 4 roots of 1)

$= 2\pi i\cdot 0$

$= 0$

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6.6 : 3, 8, 15, 24*

$z = e^{i\theta}$

$\cos\theta = \left(z + \frac{1}{z}\right)\frac{1}{2}$

$\sin\theta = \left(z - \frac{1}{z}\right)\frac{1}{2i}$

$dz = iz\,d\theta$

3 $\displaystyle\int \frac{\cos\theta}{3+\sin\theta}\,d\theta$

$$= \oint_{|z|=1} \frac{\left(z+\frac{1}{z}\right)\cdot\frac{1}{2}}{3+\frac{1}{2i}\left(z-\frac{1}{z}\right)}\,\frac{dz}{iz} = 2\pi i(1-1) = \boxed{0}$$

$$= \oint_{|z|=1} \frac{\frac{z^2+1}{z}\cdot\frac{1}{2}}{\frac{6iz+z^2-1}{2iz}}\,\frac{dz}{iz} = \oint_{|z|=1} \frac{z^2+1}{z^3+6iz^2-z}\,dz$$

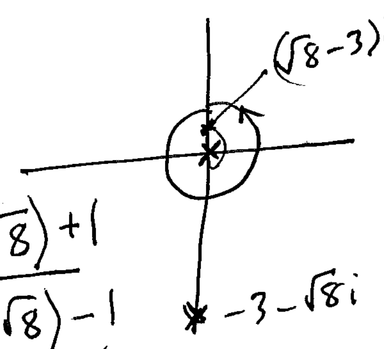
$$z^3+6iz^2-z=0 \Rightarrow z = \frac{-6i \pm \sqrt{-36-4\cdot 1\cdot -1}}{2} = \frac{-6i\pm\sqrt{-32}}{2} = -3i \pm i\sqrt{8} = (-3-\sqrt{8})i \text{ or } (2\sqrt{2}-3)i$$

$$\operatorname{Res}\left(\frac{z^2+1}{z^3+6iz^2-z}, 0, 1\right) = \lim_{z\to 0}\frac{z^2+1}{z^2+6iz-1} = -1$$

$$\operatorname{Res}\left(\frac{z^2+1}{z(z^2+6iz-1)}, (\sqrt{8}-3)i, 1\right)$$

$$= \frac{[(\sqrt{8}-3)i]^2+1}{3(\sqrt{8}-3)^2i^2+12i(\sqrt{8}-3)i-1} = \frac{(8+9-2\cdot 3\cdot\sqrt{8})+1}{-3(8+9-6\sqrt{8})-12\sqrt{8}+36-1}$$

$$= \frac{18-6\sqrt{8}}{-51+18\sqrt{8}-12\sqrt{8}+36-1} = \frac{-16+6\sqrt{8}}{-16+6\sqrt{8}} = \cancel{\frac{9-3\sqrt{8}}{-8+3\sqrt{8}}} = 1$$



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6.6.8

$$\int_0^{2\pi} \frac{\cos^2\theta}{3-\sin\theta}\,d\theta = \oint_{|z|=1} \frac{\left[\left(z+\frac{1}{z}\right)\frac{1}{2}\right]^2}{3-\left(z-\frac{1}{z}\right)\frac{1}{2i}}\,\frac{dz}{iz}$$

$\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$dz = iz\,d\theta$

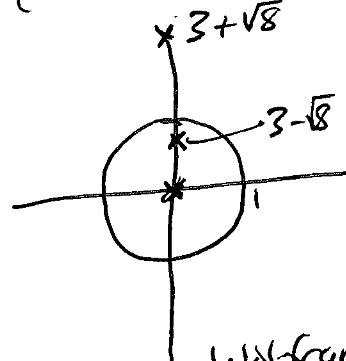
$z = e^{i\theta}$

$$I = \oint_{|z|=1} \frac{\frac{(z^2+1)^2}{4z^2}}{\frac{6iz-z^2+1}{2iz}}\cdot\frac{dz}{iz} = \int \frac{(z^2+1)^2}{-2z^4+12iz^3+2z^2}\,dz$$

$$= -\frac{1}{2}\int \frac{z^4+2z^2+1}{(z^2-6iz-1)z^2}\,dz = -\frac{1}{2}\cdot 2\pi i\cdot \text{Sum of Residues (at 0 \& } 3-2\sqrt{2})$$

$$z^2-6iz-1=0 \Rightarrow z = \frac{6i\pm\sqrt{-36+4(-1)\cdot 1}}{2} = \frac{6i\pm\sqrt{-32}}{2}$$

$$= 3i\pm i\sqrt{8} = (3+\sqrt{8})i \text{ or } (3-\sqrt{8})i$$



$$\text{Res}(f,0,2) = \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4+2z^2+1}{z^2-6iz-1}\right)$$

$$= \lim_{z\to 0}\frac{(z^2-6iz-1)\cdot(4z^3+4z)-(z^4+2z^2+1)(2z-6i)}{(z^2-6iz-1)^2}$$

$$= \frac{-1\cdot 0 - 1\cdot -6i}{(-1)^2} = 6i$$

$$= -4i\sqrt{2} \leftarrow \text{Wolfram Alpha } ☺$$

$$\text{Res}(f,(3-\sqrt{8})i,1) = \frac{\left[\left((3-\sqrt{8})i\right)^2+1\right]^2}{\left[(3-\sqrt{8})i\right]^2\cdot\left[(3-\sqrt{8})i-(3+\sqrt{8})i\right]}$$

$$= \frac{\left[-(9+8)+6\sqrt{8}+1\right]^2}{\left(-(9+8)+6\sqrt{8}\right)\left[-2\sqrt{8}i\right]} = \frac{(-16+6\sqrt{8})^2}{-96+34\sqrt{8}}i$$

$$= \frac{256+288-192\sqrt{8}}{-96+34\sqrt{8}}$$

$$= \frac{(544-384\sqrt{2})i}{68\sqrt{2}-96} = -4i\sqrt{2}$$

$$-\pi i(6i-4i\sqrt{2}) = \pi(6-4\sqrt{2}) = \pi(6-2\sqrt{8})$$

6.6.10

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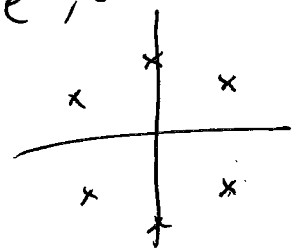
24*
$$\int_0^\infty \frac{1}{x^6+1}\,dx = \frac{1}{2}\,\text{p.v.}\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = I$$

$$\int_{-\infty}^{\infty} \frac{1}{z^6+1}\,dz = 2\pi i \,\text{Res (in upper half plane)}$$

$$z^6+1=0 \Rightarrow z = e^{\pi i/6}, e^{\frac{\pi i}{2}}, e^{\frac{5\pi i}{6}}, e^{-\frac{\pi i}{6}}, e^{-\frac{\pi i}{2}}, e^{-\frac{5\pi i}{6}}$$



$$\text{Res}\left(\frac{1}{z^6+1}, e^{\frac{\pi i}{6}}, 1\right)$$
$$= \frac{1}{6e^{\frac{5\pi i}{6}}}$$

$$\text{Res}\left(\frac{1}{z^6+1}, e^{\frac{\pi i}{2}}, 1\right) = \frac{1}{6e^{\frac{5\pi i}{2}}} = \frac{1}{6e^{\frac{\pi i}{2}}} = \frac{1}{6i} = \frac{-i}{6}$$

$$\text{Res}\left(\frac{1}{z^6+1}, e^{\frac{5\pi i}{6}}, 1\right) = \frac{1}{6e^{\frac{25\pi i}{6}}} = \frac{1}{6e^{\frac{\pi i}{6}}}$$

$$\text{Sum of residues} = \frac{1}{6}e^{-\frac{5\pi i}{6}} + \frac{1}{6}e^{-\frac{\pi i}{2}} + \frac{1}{6}e^{-\frac{\pi i}{6}}$$
$$= \frac{1}{6}\left[\cos\left(\frac{5\pi}{6}\right) - i\sin\left(\frac{\pi}{6}\right)\right] - \frac{i}{6}$$
$$+ \frac{1}{6}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right]$$
$$\frac{1}{6}\left[-\frac{i}{2} - \frac{i}{2} - i\right] = -\frac{2i}{6} = \frac{-i}{3}$$

$$\int_{-\infty}^{\infty} \frac{1}{z^6+1}\,dz = 2\pi i \cdot \frac{-i}{3} = \frac{2\pi}{3} = \int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{x^6+1}\,dx = \frac{\pi}{3}$$