Test 2: Complex Analysis

Math 312 Spring 2014
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Monday April 14
3:00-3:55pm

Name: BUCKMIRE

Directions:

Read *all* problems first before answering any of them. This tests consists of three (3) problems (and a BONUS problem) on seven (7) pages.

The topic of the problem is **in bold**, the number of points each problem is worth is in *italics* and the kind of skills required to solve each problem are in ALL CAPS.

This is a 55-minute, limited-notes*, closed book, test. **No calculators or electronic devices may be used.**

You must show all relevant work to support your answers. Use complete English sentences as much as possible and CLEARLY indicate your final answers to be graded from your "scratch work."

***You may use a one-sided 8.5" by 11" "cheat sheet" which must be stapled to the exam when you hand it in.**

Offer: If there is a formula or piece of information that you feel that you need in order to solve a problem, I will provide it to you at a non-negotiable rate of at least a one point deduction. **FEEL FREE TO ASK CLARIFICATION QUESTIONS AT ANY TIME!**

Pledge: I, ________________________________, pledge my honor as a human being and Occidental student, that I have followed all the rules above to the letter and in spirit.

No.	Score	Maximum
1		30
2		40
3		30
BONUS		10
Total		**100**

1. *[30 points.]* VERBAL, ANALYTIC. **Laurent Series, Exponential Function, Complex Powers, Cauchy Residue Theorem, Cauchy Integral Formula/e.** Consider the following statements and fill in the box with either TRUE or FALSE for each of the five statements below in this question.

To be true, the statement must ALWAYS be true. If you think the statement is FALSE, provide a counter-example of a which disprove the statement. If you think the statement is TRUE, you should provide (correct!) reasoning which proves the statement is true. You will receive *1 point* for your choice of TRUE/FALSE and *5 points* for your explanation or counterexample.

(a) *[6 pts.]* **TRUE or FALSE?** "The function $\frac{e^z - 1}{z^4}$ has a pole of order 4 at $z = 0$ with zero residue."

FALSE

$$e^z \approx 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

$$\frac{e^z - 1}{z^4} \approx \frac{1}{z^3} + \frac{1}{z^2}\frac{1}{2!} + \frac{1}{z}\frac{1}{3!} + \frac{1}{4!} + \ldots$$

This function has a pole of order 3 with residue equal to $\frac{1}{3!}$ (Coefficient of $\frac{1}{z}$ term).

(b) *[6 pts.]* **TRUE or FALSE?** "There is no solution $z \in \mathbb{C}$ to the equation $e^z + 2 = 0$."

FALSE

$$e^z = -2$$
$$z = \log(-2)$$
$$= \ln 2 + i\pi + 2k\pi i$$

There are an infinite number of solutions in the complex plane

(c) *[6 pts.]* **TRUE or FALSE?** "i^i is a purely real number; in other words the imaginary part of i^i is 0."

TRUE

$$i^i = e^{\ln(i^i)} = e^{i \ln i} = e^{i \log_e |i| + i \arg(i)}$$
$$= e^{i(0 + i\frac{\pi}{2} + 2k\pi i)}, k \in \mathbb{Z}$$
$$= e^{-(\frac{\pi}{2} + 2k\pi)}, k \in \mathbb{Z}$$

(d) *[6 pts.]* **TRUE or FALSE?** "Cauchy's Residue Theorem can be used to evaluate any contour integral."

$\boxed{\text{FALSE}}$

CRT is only useful for simple closed contours that have poles inside the contour.

See $\int_1^i \bar{z}\,dz$ from Problem 2 has nothing to do with residues

(e) *[6 pts.]* **TRUE or FALSE?** "If a function $f(z)$ is analytic everywhere on an open, connected set D, then all of its derivatives $f^{(n)}(z_0)$ exist for every $z_0 \in D$ and $n \in \mathbb{Z}^+$."

$\boxed{\text{TRUE}}$

$$f^{n}(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} dz$$ generalized CIF

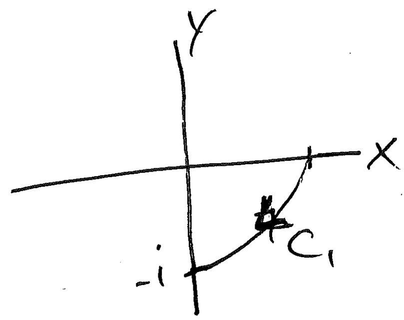
If a function is analytic on a set, it possesses infinite number of derivatives (which are also each analytic)

2. *[40 pts. total]* **Analyticity, Cauchy-Goursat Theorem, Parametrization, Contour Integration, Properties of Integrals.** VERBAL, ANALYTIC, COMPUTATIONAL, VISUAL. This problem is about evaluating

$$\mathcal{I} = \oint_C \bar{z}\, dz \qquad \text{and } \mathcal{J} = \oint_C z\, dz$$

where C is the contour formed by joining together the contours C_1 from part (a) and C_2 from part (b) given below.

(a) *[12 pts]* C_1 is the path along the arc of the unit circle from $z = 1$ to $z = -i$. Provide a sketch of the contour and the parametrizations (if any) you use to evaluate $\mathcal{I}_1 = \int_{C_1} \bar{z}\, dz$ and $\mathcal{J}_1 = \int_{C_1} z\, dz$.



$z(t) = e^{-it}, \; 0 \le t \le \frac{\pi}{2}$

$I_1 = \int_0^{\pi/2} e^{it} \cdot -ie^{-it}\, dt$

$= -i \int_0^{\pi/2} dt$

$I_1 = -\frac{\pi}{2} i$

$J_1 = \int_1^{-i} z\, dz$

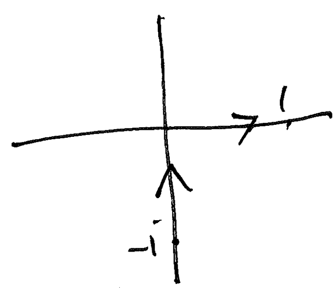
$= \frac{z^2}{2} \Big|_1^{-i} = \frac{(-i)^2}{2} - \frac{1}{2} = -\frac{1}{2} - \frac{1}{2}$

$\boxed{J_1 = -1}$

$J_1 = \int_0^{\pi/2} e^{-it} \cdot -ie^{-it}\, dt = \int_0^{\pi/2} e^{-2it}(-i)\, dt = \frac{e^{-2it}}{2}\Big|_0^{\pi/2} = -\frac{1}{2} - \frac{1}{2} = -1$

(b) *[12 pts]* C_2 consists of two straight line segments from $z = -i$ to $z = 0$ and then from $z = 0$ to $z = 1$. Provide a sketch of the contour and the parametrizations (if any) you use to evaluate $\mathcal{I}_2 = \int_{C_2} \bar{z}\, dz$ and $\mathcal{J}_2 = \int_{C_2} z\, dz$.

$J_2 = \int_{-i}^{1} z\, dz = -J_1 = \frac{z^2}{2}\Big|_{-i}^{1} = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$



$I_2 = \int_{-1}^{0} -it \cdot i\, dt + \int_0^1 t \cdot 1 \cdot dt$

$z(t) = it, \; -1 \le t \le 0$ $\qquad z(t) = t, \; 0 \le t \le 1$

$z' = i$ $\qquad z' = 1$

$I_2 = \int_{-1}^{0} t\, dt + \int_0^1 t\, dt$

$= \frac{t^2}{2}\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_0^1 = -\frac{1}{2} + \frac{1}{2} = 0$

(c) *[16 pts]* Informed by your answers to part (a) and part (b), write down the values of $\mathcal{I} = \oint_C \bar{z}\, dz$ and $\mathcal{J} = \oint_C z\, dz$. **Write a paragraph (i.e. several sentences) drawing on your understanding of the integral theorems of Cauchy to explain any relationships (or the lack of relationships) between your answers to $\mathcal{I}$ and $\mathcal{J}$, $\mathcal{I}_1$ and $\mathcal{I}_2$ and $\mathcal{J}_1$ and $\mathcal{J}_2$.**

$\bar{z}$ is NOT analytic while z is.

Therefore $J = \oint z\,dz$ must be zero by Cauchy Goursat Theorem $\boxed{J=0}$

$J_1 + J_2 = J$

Since z is analytic J_1 and J_2 are path independent, so $J_1 = -J_2 \Rightarrow J_1 + J_2 = 0$

$I_1 + I_2 = I$

But since $\bar{z}$ is not analytic I_1 & I_2 are NOT path independent

$I = 2i \cdot (\text{enclosed area}) \cdot (-1)$
$= 2i \cdot \frac{\pi}{4} \cdot -1 = \boxed{-\frac{\pi}{2} = I}$

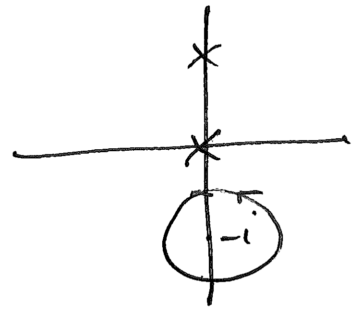
I_1 and I_2 have no relationship to each other

3. *[30 pts. total]* **Cauchy's Integral Theorems, Poles, Residues.** ANALYTICAL, VISUAL, COMPUTATIONAL. Evaluate the following integrals. **All the contours are simple, closed and traversed once in the counter-clockwise direction.**

In each problem, sketch the contour and indicate the relative location of any poles. Name any theorem you are using to justify your answer in each case.

(a) *[10 points]* Evaluate $\oint_{|z+i|=\frac{1}{2}} \frac{1}{z^2(z-i)}\, dz = A$

poles at $z = 0$ and $z = i$

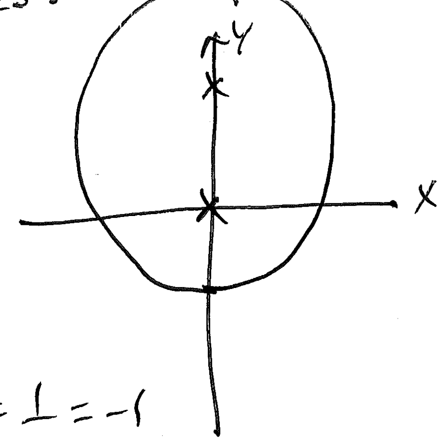


Both poles are outside the contour!

So, by Cauchy Goursat Theorem,

$$A = \oint_{(a)} dz = 0$$

(b) *[10 points]* Evaluate $\oint_{|z-i|=\frac{3}{2}} \frac{1}{z^2(z-i)}\, dz = B = 2\pi i \sum Res = 2\pi i(1-1) = 0$



$$Res\left(\frac{1}{z^2 \cdot z-i}, 0, 2\right) = \lim_{z\to 0} \frac{d}{dz}\left(\frac{1}{z-i}\right)$$

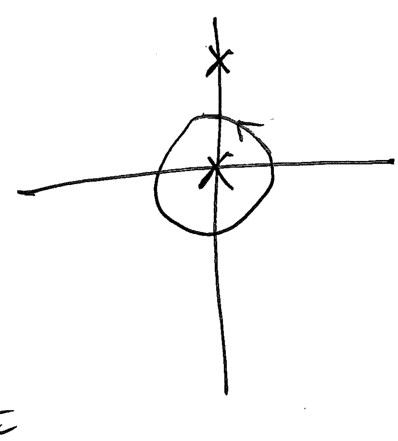
Cauchy Residue Theorem

$$= \left.-\frac{1}{(z-i)^2}\right|_{z=0}$$

$$= 1$$

$$Res\left(\frac{1}{z^2(z-i)}, i, 1\right) = \lim_{z\to i} (z-i)\frac{1}{z^2(z-i)} = \frac{1}{i^2} = \frac{1}{-1} = -1$$

(c) *[10 points]* Evaluate $\oint_{|z|=\frac{1}{2}} \frac{1}{z^2(z-i))}\, dz = C$



$$C = 2\pi i\, Res\left(\frac{1}{z^2 \cdot (z-i)}, 0, 2\right)$$

$$= 2\pi i \cdot (1)$$

$$= 2\pi i$$

Cauchy Residue Theorem or CIF

BONUS *[10 pts.]* **Application of Contour Integration To Real Integrals, Residues.**
By evaluating an appropriate contour integral show that

$$I = \int_0^\infty \frac{dx}{(x^2+1)(x^2+4)} = \frac{\pi}{12} \qquad \text{OR} \qquad \int_0^{2\pi} \frac{2d\theta}{2+\cos\theta} = \frac{4\pi}{\sqrt{3}} = J$$

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\oint_C \frac{dz}{(z^2+1)(z^2+4)}$$

$$= \frac{1}{2}\cdot 2\pi i\left[\text{Res}\left(\frac{1}{(z^2+1)(z^2+4)}, i\right) + \text{Res}\left(\frac{1}{(z^2+1)(z^2+4)}, 2i\right)\right]$$

$$\text{Res}\left(\frac{1}{(z^2+1)(z^2+4)}, i\right) = \lim_{z\to i} \frac{\cancel{(z-i)}}{(z+i)\cancel{(z-i)}(z^2+4)}$$

$$= \frac{1}{2i\cdot 3} = \frac{1}{6i}$$

$$\text{Res}\left(\frac{1}{(z^2+1)(z^2+4)}, 2i\right) = \lim_{z\to 2i} \frac{\cancel{z-2i}}{(z+2i)\cancel{(z-2i)}(z^2+1)}$$

$$= \frac{1}{4i}\cdot\frac{1}{-3} = \frac{-1}{12i}$$

$$I = \frac{1}{2}\cdot 2\pi i\left(\frac{1}{6i} - \frac{1}{12i}\right) = \pi i\left(+\frac{1}{12i}\right) = \frac{\pi}{12}$$

$$J = \oint_{|z|=1} \frac{2}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz} = \oint \frac{2}{\frac{4z+z^2+1}{2z}}\frac{dz}{iz}$$

$z = e^{i\theta}$
$dz = iz\,d\theta$

$$= \frac{4}{i}\oint \frac{dz}{z^2+4z+1} \qquad z = \frac{-4 \pm \sqrt{16-4\cdot 1\cdot 1}}{2}$$

$$= \frac{4}{i}\cdot 2\pi i\cdot\text{Res}\left(\frac{1}{z^2+4z+1}, -2+\sqrt{3}\right) \qquad = -2\pm\sqrt{3}$$

$= -2-\sqrt{3}$ or $-2+\sqrt{3}$

inside $|z| \le 1$

$$= 8\pi\cdot\frac{1}{2(-2+\sqrt{3})+4}$$

$$= \frac{8\pi}{-4+2\sqrt{3}+4} = \frac{8\pi}{2\sqrt{3}} = \frac{4\pi}{\sqrt{3}}$$