1. *[28 pts. total]* **Analyticity, Cauchy-Riemann Equations, Cauchy-Schwarz Inequality, Complex Functions of a Complex Variable.** ANALYTIC, COMPUTATIONAL, VERBAL. Determine whether the following statements are **TRUE** or **FALSE** and place your answer in the box. To receive FULL credit, you must also give a very brief (and correct) explanation in support of your TRUE/FALSE choice! The explanation for your answer is worth SIX (6) POINTS while your correct TRUE or FALSE answer is worth ONE (1) point.

(a) *[7 points]*. **TRUE or FALSE**: "The function $f(z) = x^2 + y^2 + 2xyi$ is not analytic at any point in the Complex plane."

$\boxed{\text{TRUE}}$

$u = x^2 + y^2$ $\quad u_x = 2x$ $\quad u_y = 2y$

$v = 2xy$ $\quad v_x = 2y$ $\quad v_y = 2x$

CREs

$2x = 2x$ ✓

$2y = -2y \Rightarrow y = 0$

$f(z)$ is differentiable at $y = 0$ but this is not an open set so f is analytic nowhere.

(b) *[7 points]*. **TRUE or FALSE**: "The function Arg(z) maps the entire complex plane to a subset of the real numbers."

$\boxed{\text{FALSE}}$

0 is not mapped by Arg z. so it's not the entire complex plane.

It is true that the output of Arg z is a range of real numbers $-\pi < \text{Arg}\, z \leq \pi$.

(c) *[7 points]*. **TRUE or FALSE**: "If $|z| \leq 2$, then $|z^2 - 2iz - 1| \leq 9$."

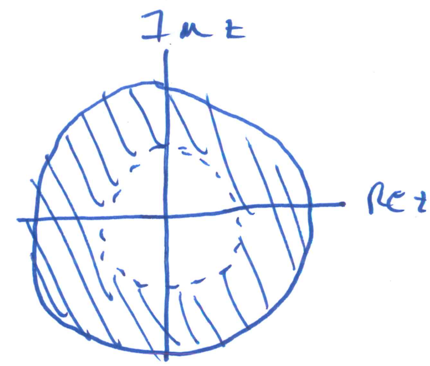
$\boxed{\text{TRUE}}$

Use Triangle Inequality

$$|z^2 - 2iz - 1| \leq |z^2| + |-2iz| + |-1|$$
$$\leq |z|^2 + 2 \cdot |z| + 1$$
$$\leq 2^2 + 2 \cdot 2 + 1$$
$$\leq 9 \checkmark$$

(d) *[7 points]*. **TRUE or FALSE**: "The set of points $A = \{z \in \mathbb{C} : 1 < |z| \leq 2\}$ is both open and closed."

$\boxed{\text{FALSE}}$



This set is neither open or closed. It does not contain all its boundary points so it's not closed and it contains some boundary points so it's not open!

2. *[32 pts. total]* **Operations on Complex Numbers, Argand Plane, Visualization.** VISUAL, ANALYTIC, COMPUTATIONAL, VERBAL. Consider an unspecified complex number $Z = a(1+i)$, written as (x, y) coordinates of a point in the Argand plane as $Z = (a, a)$ where a **is a specific (fixed) positive real number such that** $2 < a < 3$.

On each of the axes below, sketch the associated complex number W in the complex plane. (You will get 4 points for drawing the correct geometric location of W and Z, 2 points for the correct coordinates of W in each case written in the box and 2 points for a short explanation and/or illustrative work which supports your answers).

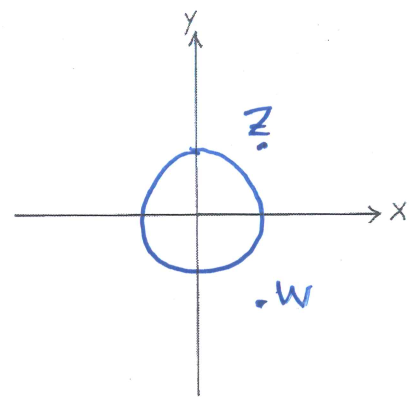
ON EACH FIGURE DRAW THE LOCATION OF W <u>AND</u> Z.
ADD THE UNIT CIRCLE CENTERED AT THE ORIGIN FOR SCALE.
WRITE THE COORDINATES FOR W IN THE BOX.

(a) *[8 points]* $W = \overline{Z}$ $\boxed{(a, -a)}$

$W = \overline{a(1+i)} = \overline{a + ai}$

$= a - ai$

W represents a reflection across the x-axis

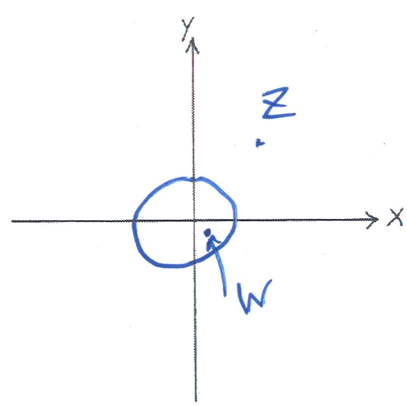


(b) *[8 points]* $W = \dfrac{1}{Z} = \dfrac{\overline{Z}}{|Z|^2}$ $\boxed{(1/2a, -1/2a)}$

$W = \dfrac{1}{a + ai} = \dfrac{a - ai}{a^2 + a^2}$

$= \dfrac{a(1-i)}{2a^2}$

$= \dfrac{1}{2a}(1-i)$



W is on the y = -x line but inside the unit circle

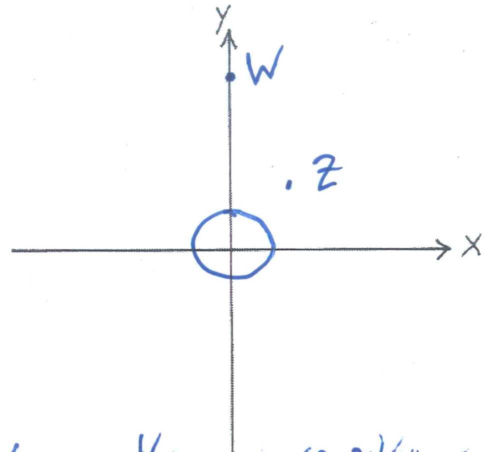
ON EACH FIGURE DRAW THE LOCATION OF W AND Z.
ADD THE UNIT CIRCLE CENTERED AT THE ORIGIN FOR SCALE.
WRITE THE COORDINATES FOR W IN THE BOX.

(c) *[8 points]* $W = Z^2$

$\boxed{(0, 2a^2)}$

$$W = (a+ai)^2$$
$$= a^2(1+i)^2$$
$$= a^2(1+2i-1)$$
$$= 2a^2 i$$

W is purely imaginary



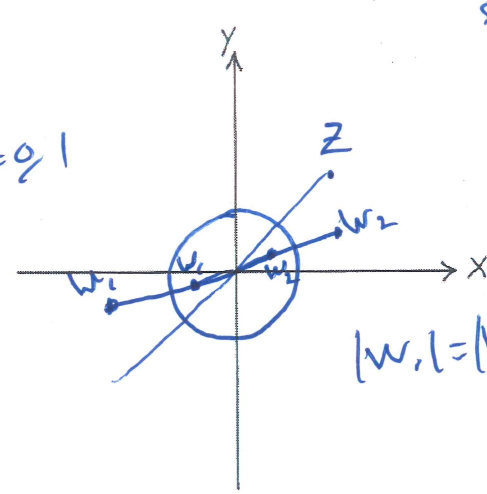
Cartesian coordinates $W_2 = \left((2a^2)^{1/4}\cos\frac{9\pi}{8}, (2a^2)^{1/4}\sin\frac{9\pi}{8}\right)$, $\left((2a^2)^{1/4}\cos \pi/8, (2a^2)^{1/4}\sin \pi/8\right) = W_1$

(d) *[8 points]* $W^2 = Z$

$\boxed{\left((2a^2)^{1/4}, \pi/8\right) \text{ or } \left((2a^2)^{1/4}, -\frac{7\pi}{8}\right)}$

exponential or polar form suffices

$$W^2 = a(1+i)$$
$$W = a^{1/2}(1+i)^{1/2}$$
$$(1+i)^{1/2} = |1+i|^{1/2} e^{i\frac{(Arg(1+i)+2\pi k)}{2}}, \quad k = 0, 1$$
$$= (\sqrt{2})^{1/2} e^{i(\pi/4 + 2k\pi)\frac{1}{2}}$$
$$= 2^{1/4} e^{i\frac{\pi}{8}(1+8k)}, \quad k = 0, 1$$
$$= 2^{1/4} e^{i\pi/8}, \; 2^{1/4} e^{\frac{9\pi i}{8}}$$
$$= 2^{1/4} e^{i\pi/8}, \; 2^{1/4} e^{-7\pi i/8}$$
$$W = (2a^2)^{1/4} e^{i\pi/8}, \; (2a^2)^{1/4} e^{-\frac{7\pi i}{8}}$$



$|W_1| = |W_2| > 1$

3. *[40 pts. total]* **Mapping, Reciprocal Function, Composition, Point Sets in the Complex Plane.** ANALYTICAL, VISUAL, COMPUTATIONAL. One of the most important class of mappings in Complex Analysis is the class of functions called linear fraction transformations (LFT). $w = B(z) = \dfrac{1-z}{1+z}$ is an example of an LFT. Every LFT can be thought of as a composition of two linear mappings and a reciprocal mapping.

(a) *[4 pts]* Use algebra to show that $B(z) = \dfrac{1-z}{1+z} = -1 + 2\dfrac{1}{1+z}$.

$$\text{RHS} = -1 + 2\left(\frac{1}{1+z}\right) = \frac{-(1+z)+2}{1+z}$$

$$= \frac{-1-z+2}{1+z} = \frac{1-z}{1+z}$$

(b) *[4 pts]* Given $f_3(z) = -1 + 2z$, $f_2(z) = \dfrac{1}{z}$ and $f_1(z) = 1 + z$ use algebra to **show that** $B(z)$ **can be written as the composition of these two linear mappings and the reciprocal mapping**, i.e. $B(z) = (f_3 \circ f_2 \circ f_1)(z)$ or $B(z) = f_3(f_2(f_1(z)))$.

$$f_3(f_2(f_1(z))) = f_3(f_2(1+z)) = f_3\left(\frac{1}{1+z}\right)$$

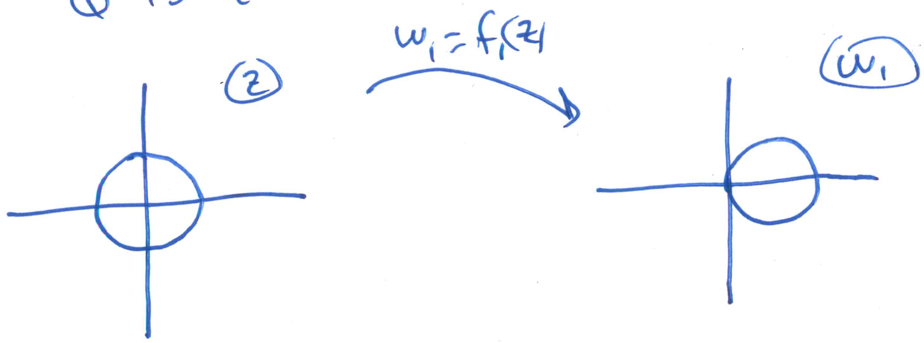
$$= -1 + 2\left(\frac{1}{1+z}\right)$$

(c) *[4 pts]* Show that the the image of the set $\mathcal{P} = \{z \in \mathbb{C} \ : \ |z| = 1\}$ under the mapping $w_1 = f_1(z)$ is the set $\mathcal{Q} = \{w_1 \in \mathbb{C} \ : \ |w_1 - 1| = 1\}$. **Describe the pre-image** $\mathcal{P}$ **and image** $\mathcal{Q}$ **in words** and **categorize in general terms the effect** f_1 **has on a set of points** (i.e. as some combination of rotation, scaling, translation).

$f_1(z)$ is a TRANSLATION of 1 unit to the right

P is a circle of radius 1 at the origin

Q is a circle of radius 1 at $(1,0)$.



(d) *[8 pts]* Recall that the inversion mapping $w = \dfrac{1}{z}$ maps lines given by $\mathrm{Re}(z) = k$ to circles given by $\left|w - \dfrac{1}{2k}\right| = \left|\dfrac{1}{2k}\right|$ for all $k \neq 0$. Using this information and the idea that the reciprocal function is its own inverse, **what is the image $\mathcal{R}$ of the set $\mathcal{Q} = \{w_1 \in \mathbb{C}$: $|w_1 - 1| = 1\}$ under the mapping $w_2 = f_2(w_1) = \dfrac{1}{w_1}$? Describe $\mathcal{R}$ in point set notation.**

$|w_1 - 1| = 1$ is Q

$\left|w_1 - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right| \Rightarrow \frac{1}{2k} = 1 \Rightarrow k = \frac{1}{2}$

So the inversion mapping will map circles of $\left|w - \frac{1}{2k}\right| = \frac{1}{2k}$ to line $\mathrm{Re}(z) = \cancel{k} = \frac{1}{2}$

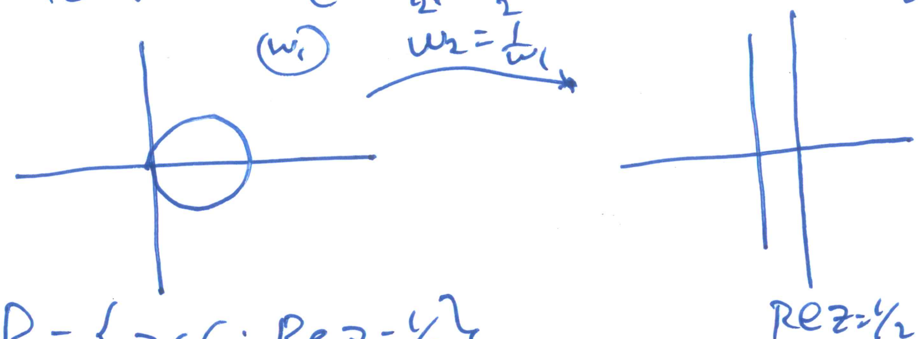
map points

$0 \xrightarrow{w = \frac{1}{z}} \infty$

$1 + i \longrightarrow \frac{1}{1+i} = \frac{1}{2} - \frac{i}{2}$

$2 \longrightarrow \frac{1}{2}$

must be line



$R = \{z \in \mathbb{C} : \mathrm{Re}\, z = 1/2\}$

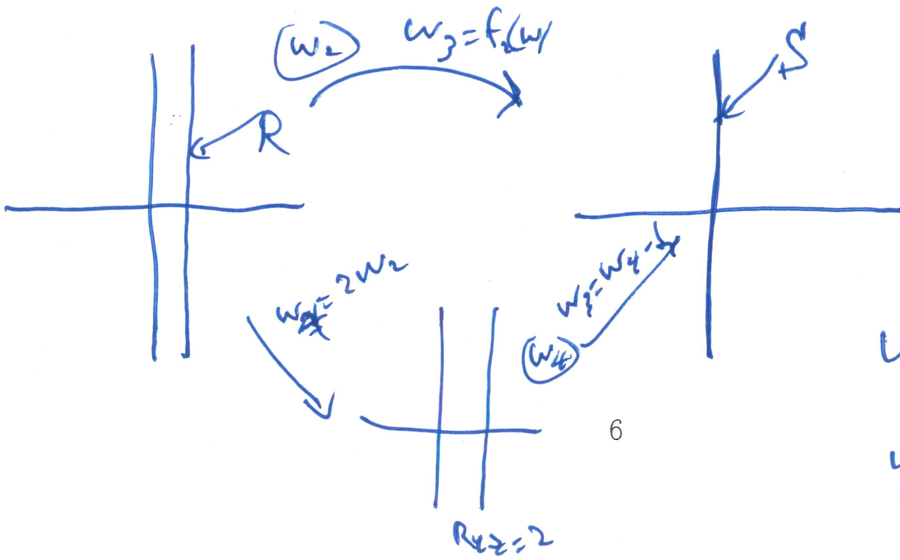
(e) *[4 pts]* Using your answer from **(d)**, show that $\mathcal{S} = \{w \in \mathbb{C}$: $\mathrm{Re}(w) = 0\}$ is the image of $\mathcal{R}$ under the mapping $w = f_3(w_2) = -1 + 2w_2$. **Describe the pre-image $\mathcal{R}$ and image $\mathcal{S}$ in words** and **categorize in general terms the effect f_3 has on a set of points** (i.e. as some combination of rotation, scaling, translation).

R is the vertical line at $x = 1/2$

S is the y-axis.

moves line to $\mathrm{Re}\, z = 1$

f_3 scales w_2 making it twice as big and then translates it 1 unit to the left



parametrize R

$z = \frac{1}{2} + it, \ -\infty < t < \infty$

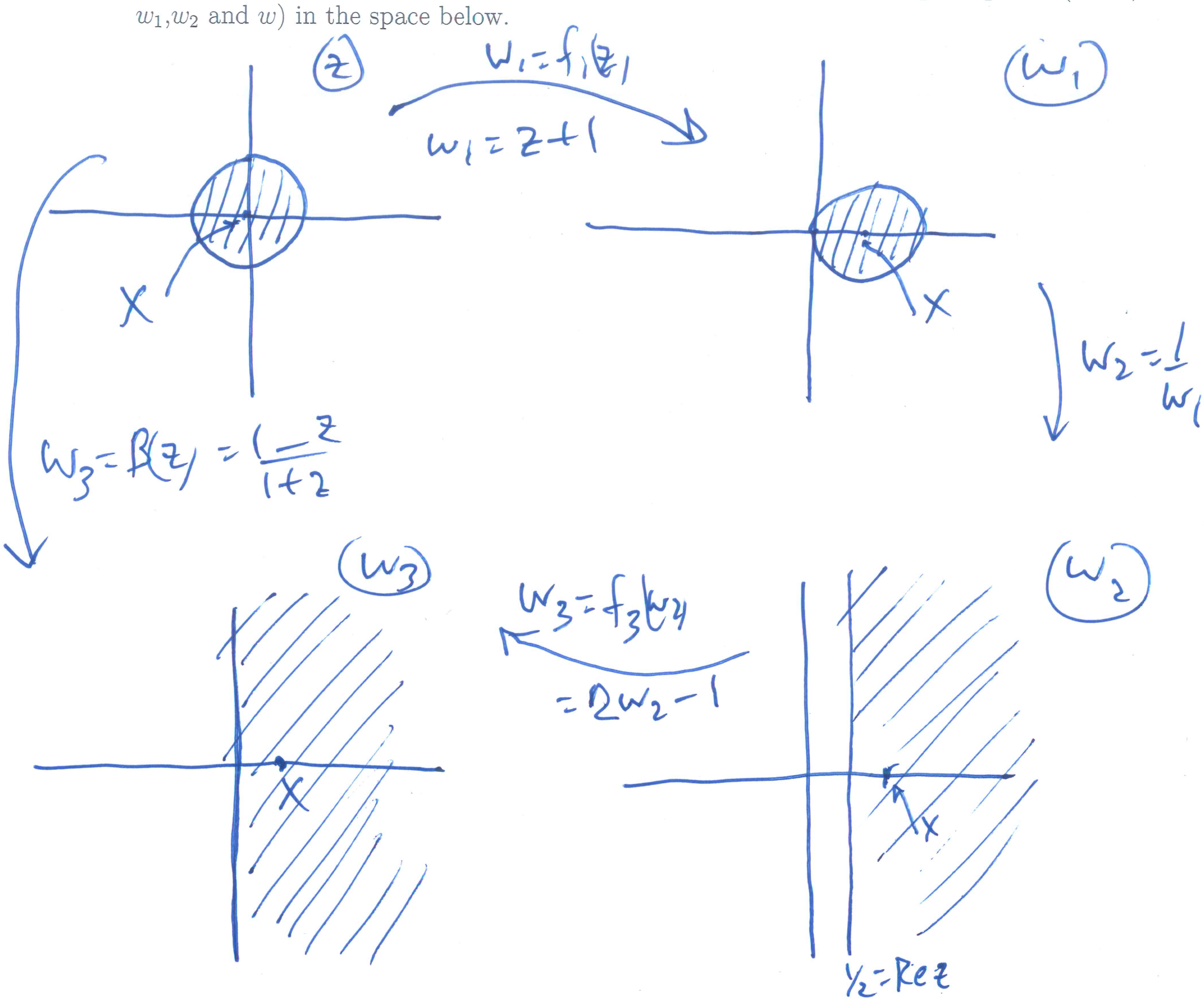
$w = -1 + 2\left(\frac{1}{2} + it\right)$

$= -1 + 1 + 2it$

$w = 2it, \ -\infty < t < \infty$

represents the Imaginary axis

(f) *[12 pts]* Draw pictures which demonstrate the action of $w = B(z) = \dfrac{1-z}{1+z}$ on the set of points $|z| = 1$ in the space below by showing the action of each constituent mapping (i.e. f_1 then f_2 then f_3 on $|z| = 1$). I expect you to do this by **sketching the curves $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$ and $\mathcal{S}$ on four separate, clearly labelled copies of the complex plane** (i.e. z, w_1,w_2 and w) in the space below.



$B(0) = \dfrac{1-0}{1+1} = 1$

(g) *[4 pts]* **Indicate on your sketch above where the interior of the unit disk** $|z| \le 1$ **gets mapped to under** $B(z)$. (HINT: Pick a point $\mathcal{X}$ inside $|z| \le 1$ and show where it moves under each constituent mapping f_1 then f_2 then f_3 from the z-plane to the w_1-plane to the w_2-plane to the w-plane on your four axes drawn above.)

BONUS *[5 pts.]*

This BONUS problem is about deriving the Cauchy-Riemann Equations in polar coordinates. Consider $f(z) = u(x,y) + iv(x,y)$ where $x = r\cos\theta$ and $y = r\sin\theta$ so that $f(z) = u(r,\theta) + iv(r,\theta)$.

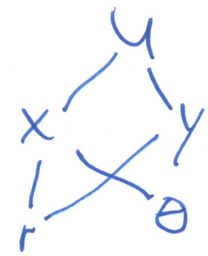
(a) *[2 BONUS POINTS.]* Use the multivariable chain rule to obtain expressions for u_r in terms of u_x and u_y, v_r in terms of v_x and v_y, u_θ in terms of u_x and u_y, and v_θ in terms of v_x and v_y.

(b) *[3 BONUS POINTS.]* By substituting the equations $u_x = v_y$, $\quad u_y = -v_x$ into your expressions for v_r and v_θ and comparing the results with your expressions for u_r and u_θ you should be able to deduce the Cauchy-Riemann Equations in polar coordinates.

(a) $u(x,y) = u(r\cos\theta, r\sin\theta)$ $\qquad x = r\cos\theta \quad y = r\sin\theta$

$u_r = u_x x_r + u_y y_r$ $\qquad x_r = \cos\theta \quad y_r = \sin\theta$

$u_\theta = u_x x_\theta + u_y y_\theta$ $\qquad x_\theta = -r\sin\theta \quad y_\theta = r\cos\theta$



$v_r = v_x x_r + v_y y_r$

$v_\theta = v_x x_\theta + v_y y_\theta$

(b) $u_r = u_x \cos\theta + u_y \sin\theta$

$u_\theta = u_x(-r\sin\theta) + u_y(r\cos\theta) =$

$v_r = v_x \cos\theta + v_y \sin\theta = -u_y\cos\theta + u_x \sin\theta$

$v_\theta = -v_x r\sin\theta + r v_y \cos\theta = u_y r\sin\theta + u_x r\cos\theta$

$= r(u_y \sin\theta + u_x\cos\theta)$

$v_\theta = r u_r$

$r v_r = -u_y r\cos\theta + u_x r\sin\theta = -u_\theta$

$= -(u_y r\cos\theta - u_x r\sin\theta)$

$$\left\{\begin{array}{l} v_r = -\frac{1}{r} u_\theta \\ \frac{1}{r} v_\theta = u_r \end{array}\right\}$$ CREs in polar form!