BUCKMIRE

Complex Analysis

Math 312 Fall 2001

M 5-6:25, R 1:30-2:55
Fowler 112, Fowler 201

TEST 1: Friday October 19, 2001

Directions: Read *all* 4 problems first before answering any. **You may choose to answer question 3 or question 4.** You may have access to any notes or the textbook. This is a one hour test. You have 90 minutes to complete it.

No.	Score	Maximum
1		40
2		30
3 or 4		30
Total		100

1. *[40 pts. total]* **Mapping.** We want to find the general form of the function $M(z) = Az+B$ which maps one circle, the set of points **P**: $|z - z_0| = r$, to another circle located somewhere else, the set of points **Q**: $|w - w_0| = \rho$ in the complex plane.

(a) *[5 pts]* Find a mapping of the form $f_1(z) = \alpha z$ which maps **P** so that it has the same radius as **Q**.

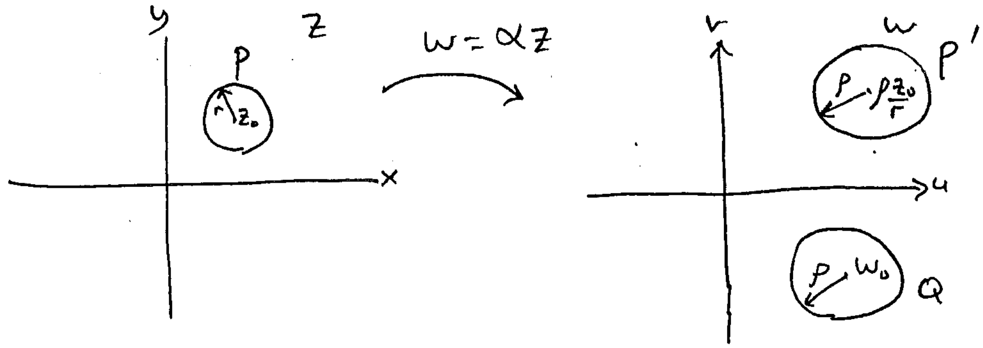


Image of P under $f_1 = \alpha\cdot$ is circle of radius $\alpha r = \rho$ centered at αz_1

$w = \alpha z$

$\frac{w}{\alpha} = z$ $\quad |z - z_0| = r \Rightarrow \left|\frac{w}{\alpha} - z_0\right| = r$

$|w - \alpha z_0| = \alpha r$

$\boxed{\alpha = \frac{\rho}{r}}$

(b) *[5 pts]* Find a mapping of the form $f_2(z) = z + \beta$ which maps **P** so that its center is at the same location as **Q**.

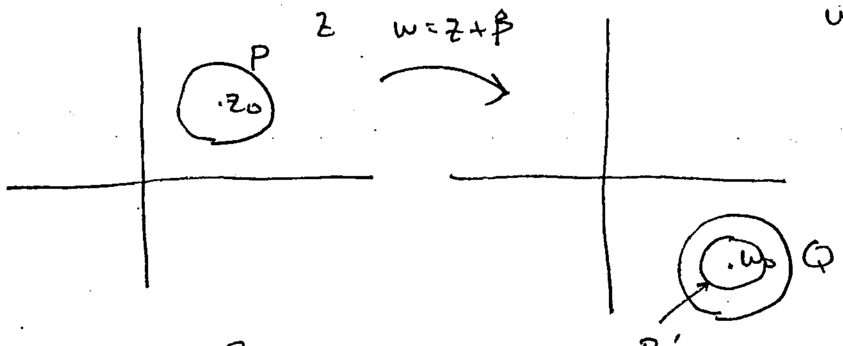


Image of P under $f_2 = z + \beta$ is circle of di r located at

$f_2(z) = z + w_0 -$

$w = z + \beta$

$w - \beta = z$ $\quad |z - z_0| = r \Rightarrow |w - \beta - z_0| = r$

we want $\beta + z_0 = w_0 \Rightarrow \boxed{\beta = w_0 - z}$

(c) *[10 pts]* Will $f_2(f_1(z)) = F(z)$ be the mapping which maps **P** to **Q**? In other words, what is the image of **P** under $F(z)$?

$w = F(z) = f_2(f_1(z)) = f_2\left(\frac{\rho}{r} z\right) = \frac{\rho}{r} z + w_0 - z_0 = w$

$z + \frac{r}{\rho}(w_0 - z_0) = \frac{r}{\rho} w \Rightarrow z = \frac{r}{\rho} w - \frac{r}{\rho}(w_0 - z_0)$

$|z - z_0| = r \Rightarrow \left|\frac{r}{\rho} w - \frac{r}{\rho}(w_0 - z_0)\right| = r$

$\Rightarrow |w - (w_0 - z_0)| = \frac{\rho}{r} \cdot r = \rho$

Image of $|z - z_0| = r$ under $w = F(z)$ is $|w - (w_0 - z_0)| = \rho$

It has the correct RADIUS but incorrect LOCATION

(d) *[10 pts]* Find an example of a mapping $M(z) = Az + B$ where A and B depend on the parameters r, ρ, z_0 and w_0 which maps **P** to **Q**.

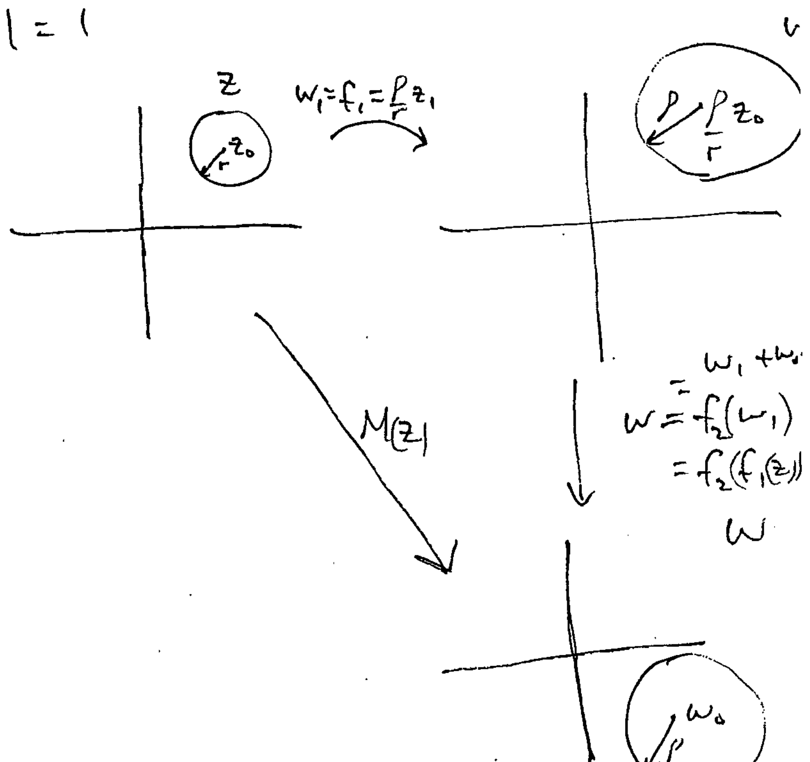
$$|z - z_0| = r \xrightarrow{M(z)} |w - w_0| = \rho$$

$$\left|\frac{z}{r} - \frac{z_0}{r}\right| = 1 \qquad \left|\frac{w}{\rho} - \frac{w_0}{\rho}\right| = 1$$

$$\frac{z - z_0}{r} = \frac{w - w_0}{\rho}$$

$$\frac{\rho}{r}(z - z_0) = w - w_0$$

$$\boxed{w_0 + \frac{\rho}{r}(z - z_0) = w}$$



$$w_0 - \frac{\rho}{r} z_0 + \frac{\rho}{r} z = w$$

$$B + Az = w$$

$$w = \frac{\rho}{r} \cdot z + w_0 - \frac{\rho}{r} z_0$$

SCALE by $\frac{\rho}{r}$ THEN Translate by $w_0 - \frac{\rho}{r} z_0$

So $f_1 = \frac{\rho}{r} z$, $f_2 = z + w_0 - \frac{\rho}{r} z_0$

$M(z) = f_2(f_1(z))$

(e) *[10 pts]* Use your answers above to find the function $M(z)$ which maps $|z - 2 - i| = 1$ to $|w + 2 + 3i| = 2$.

$z_0 = 2 + i \qquad r = 1$

$w_0 = -2 - 3i \qquad \rho = 2$

$$M(z) = -2 - 3i - \frac{2}{1}(2 + i) + \frac{2}{1} z$$

$$= -2 - 3i - 4 - 2i + 2z$$

$$w = M(z) = -6 - 5i + 2z$$

$w = -6 - 5i + 2z$

$\frac{w + 6 + 5i}{2} = z$

$1 = |z - 2 - i| = \left|\frac{w}{2} + 3 + \frac{5i}{2} - 2 - \right.$

$1 = \left|\frac{w}{2} + 1 + \frac{3i}{2}\right|$

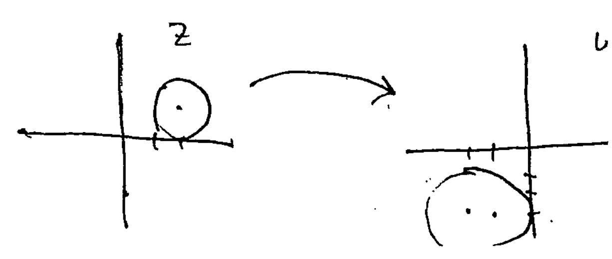
$2 = |w + 2 + 3i|$

$M(2+i) = -2 - 3i - 2(2+i) + 2(2+i)$

$= -2 - 3i \leftarrow$ center of Q

$\uparrow$

center of P



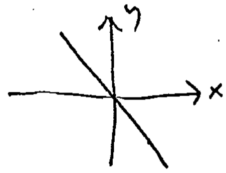
2. *[30 pts.]* **Arithmetic of Complex Numbers.** **(a)** *[10 pts]* What condition on a and b must be met for $(a+bi)^2 = ci$ where a, b and c are all real numbers? Where in the complex plane would (a,b) have to be for c to be negative?

$a^2 - b^2 + 2abi = ci$

$a^2 - b^2 = 0$ $\quad a^2 = b^2 \Rightarrow$ either $a = b$ or $a = -b$

$2ab = c$ $\quad$ If $c < 0$

$a = -b$



The points (a,b) are in 2^{nd} or 4^{th} quadrant

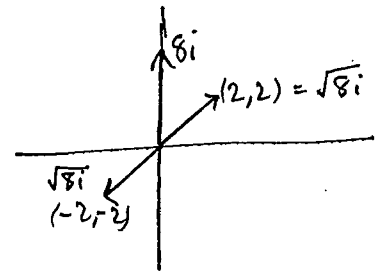
(b) *[10 pts]* Use your answer in part **(a)** to help you evaluate $\sqrt{8i}$.

$a^2 - b^2 = -8$

$2ab = +8$

$a = b$

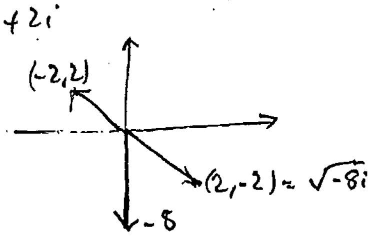
$a = \pm 2 \quad b = \pm 2$



$\sqrt{8i} \quad a = 2 + 2i, \ -2 - 2i$

$\sqrt{-8i} = \sqrt{-1} \cdot \sqrt{8i} = i \cdot (2+2i) = -2 + 2i$

$i(-2-2i) = 2 - 2i$



(c) *[10 pts]* Use your answers above to find all the solutions of $z^2 + 2iz + 8i - 1 = 0$.

$$z = \frac{-2i \pm \sqrt{-4 \pm 4(+1)(8i-1)}}{2(+1)}$$

$$= \frac{-2i \pm \sqrt{-4 - 32i + 4}}{2(1)} = \frac{-2i \pm \sqrt{-32i}}{+2}$$

$= -i \pm \sqrt{-8i}$

$= -i \pm -2 + 2i$

$= -i + -2 + 2i$

OR

$-i - (-2 + 2i)$

$= -2 + i$ OR $2 - 3i$

3. *[30 pts. total]* **Cauchy-Riemann Equations, Harmonic Functions.** Consider the function $f = u(x,y) + iv(x,y) = x^2 + y^2 + 2xyi$

(a) *[10 pts]* Show that the set of points for which $f'(z)$ exists all lie on the x-axis.

$u_x = 2x \qquad u_y = 2y$

$V_x = 2y \qquad V_y = 2x$

CREs

$u_x = V_y \qquad 2x = 2x$ true for all x

$u_y = -V_x \qquad 2y = -2y$ true when $y=0$

(b) *[10 pts]* Using your information from (a), on what set of points is $f(z)$ analytic? Explain your answer.

The derivative exists ONLY at x any, $y=0$. The x-axis is NOT an open set, so $f(z)$ is ANALYTIC NOWHERE

(c) *[10 pts]* Show that $v(x,y)$ is harmonic. Is the given $u(x,y)$ its harmonic conjugate? If not, find the harmonic conjugate of $v(x,y)$

$V_x = 2y \qquad V_y = 2x \qquad\qquad u_x = 2x \qquad u_y = 2y$

$V_{xx} = 0 \qquad V_{yy} = 0 \qquad\qquad u_{xx} = 2 \qquad u_{yy} = 2$

$V_{xx} + V_{yy} = 0 \qquad\qquad u_{xx} + u_{yy} = 4 \neq 0$

V is harmonic $\qquad\qquad$ u is NOT harmonic!

$u(x,y) = x^2 - y^2$ is the harmonic conjugate of $V(x,y) = 2xy$

$V_x = 2y = -u_y \qquad u_y = -2y \qquad\qquad u(x,y) = x^2 - y^2 + C$

~~$V(x,y) = y^2$~~

$u = -y^2 + A(x)$

$u_x = V_y \Rightarrow A'(x) = 2x$

$A(x) = x^2 + C$

4. *[30 pts. total]* **Analyticity, Differentiability.**

The **Jacobian** of a mapping $u = u(x,y), v = v(x,y)$ from the xy-plane to the uv-plane is defined to be the determinant

$$J(x_0, y_0) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix},$$

where the partial derivatives u_x, u_y, v_x, v_y are all evaluated at (x_0, y_0).

(a) *[10 pts]* If $f = u + iv$ is analytic on a neighborhood containing $z_0 = x_0 + iy_0$ show that $J(x_0, y_0) = |f'(z_0)|^2$.

$J = u_x v_y - u_y v_x$

If f is analytic, f' exists and thus CRE's are true.

$u_x = v_y \quad u_y = -v_x$

$J = u_x \cdot u_x - (-v_x) v_x = u_x^2 + v_x^2 = |u_x + i v_x|^2$

$= |f'(z_0)|^2$

(b) *[20 pts]* For the function $f(z) = Az + B$ find $J(0,0)$ two different ways (i.e. from the definition and from the result given in part **(a)**.

Assume A, B real

$f = Az + B = A(x+iy) + B$

$u(x,y) = Ax + B \qquad v(x,y) = \cancel{i}Ay$

$u_x = A \qquad v_x = 0$

$u_y = 0 \qquad v_y = A$

$J(0,0) = \begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix} = A^2$

$f'(z) = A$

$J(0,0) = |A|^2 = A^2$

$A, B \in \mathbb{C} \quad f = (A_1 + iA_2)(x+iy) + (B_1 + iB_2)$

$= A_1(x+iy) + iA_2(x+iy) + B_1 + iB_2$

$= A_1 x - \cancel{i}A_2 y + B_1 + i(A_1 y + A_2 x + B_2)$

$u_x = A_1 \quad u_y = -A_2$

$v_x = A_2 \quad v_y = A_1$

$J(0,0) = \begin{vmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{vmatrix}$

$= A_1^2 + A_2^2$

$= |A_1 + iA_2|^2$

$= |A|^2$