Complex Analysis

Math 214 Spring 2014 | Fowler 307 MWF 3:00pm - 3:55pm

 | http://faculty.oxy.edu/ron/math/312/14/

Class 30: Monday April 21

TITLE Return To Laurent Series
CURRENT READING Zill & Shanahan, §6.1-6.2
HOMEWORK Zill & Shanahan, §6.2: 3,15,20,24. §6.3 7,8,9,10.

SUMMARY
We shall return to our study of Laurent Series for functions with singularities (simple poles). We can find Laurent Series expansions for such function in the annular regions between the poles. We will use an applet written by Terence Tao to help explore this topic.

Today we will be exploring Laurent Series with the help of Terence Tao's Java Applet on Laurent Series at

`http://www.math.ucla.edu/~tao/java/Laurent.html`

Consider the function $f(z) = \dfrac{1}{(z-1)(z-2)}$.

The poles of $f(z)$ are at $z = 1$ and $z = 2$. Recall that the MacLaurin Series for $f(z) = \dfrac{1}{1-z}$ is $1 + z + z^2 + z^3 + \ldots$ or $\displaystyle\sum_{k=0}^{\infty} z^k$ which converges when $|z| < 1$

Exercise

Show that $\dfrac{1}{(z-1)(z-2)}$ can be written as $\dfrac{1}{1-z} + \dfrac{1}{z-2}$.

Also show that $\dfrac{1}{z-2} = -\dfrac{1}{2}\left(\dfrac{1}{1-\frac{z}{2}}\right)$

EXAMPLE

We can use the above two facts to show that the MacLaurin Series expansion for $f(z) = \dfrac{1}{(z-1)(z-2)}$ is $\dfrac{1}{2} + \dfrac{3}{4}z + \dfrac{7}{8}z^2 + \dfrac{15}{16}z^3 + \ldots = \displaystyle\sum_{k=0}^{\infty} \frac{2^{k+1}-1}{2^{k+1}} z^k$ valid when $|z| < 1$

We end up using the fact that

$$\frac{1}{z-2} = -\sum_{k=0}^{\infty} 2^{-k-1} z^k \qquad \text{valid when } |z| < 2 \tag{1}$$

Laurent series

A **Laurent Series** is used to expand a function about a point in regions between where it has a singularity. For $f(z) = \dfrac{1}{(z-1)(z-2)}$ there are simple poles at $z = 1$ and $z = 2$ so we can find a Laurent Series in the region $1 < |z| < 2$ and another in the region $|z| > 2$. We already know this function has a MacLaurin Series that converges to $f(z)$ for $|z| < 1$.

If we use the idea that $\dfrac{1}{1-\Box}$ is equal to $1 + \Box + \Box^2 + \Box^3 + \ldots = \sum_{k=0}^{\infty} \Box^k$ when $|\Box| < 1$

$$\begin{aligned} \frac{1}{z-1} &= \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \\ &= \sum_{k=0}^{\infty} z^{-k-1} \end{aligned} \tag{2}$$

Note this series is convergent when $\left|\frac{1}{z}\right| < 1$ or $|z| > 1$.

We can use a similar process to get a Laurent Series expansion for $\dfrac{1}{z-2}$ about $z = 0$.

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k = \sum_{k=0}^{\infty} 2^k z^{-k-1} \qquad \text{valid when } |z| > 2 \tag{3}$$

This second expansion is only valid in the region where $\left|\frac{2}{z}\right| < 1$ or $|z| > 2$

In order to obtain the Laurent Series expansion for $\dfrac{1}{(z-1)(z-2)}$ about $z = 0$ in the region $1 < |z| < 2$ we need the Laurent Series for $\dfrac{1}{z-2}$ in the region $|z| < 2$ [i.e. equation (1)] and the Laurent Series for $\dfrac{1}{z-1}$ in the region $|z| > 1$ [i.e. equation (2)].

$$\begin{aligned} \frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = -\sum_{k=0}^{\infty} 2^{-k-1} z^{k-} - \sum_{k=0}^{\infty} z^{-k-1} \\ &= \ldots - \frac{1}{z^2} - \frac{1}{z} - \frac{1}{2} - \frac{z}{4} - \ldots \end{aligned}$$

while in the region $|z| > 2$ the Laurent Series for $\dfrac{1}{(z-1)(z-2)}$ about $z = 0$ is given by the Laurent Series for $\dfrac{1}{z-2}$ in the region $|z| > 2$ [i.e. equation (3)] and the Lauren Series for $\dfrac{1}{z-1}$ in the region $|z| > 1$ [i.e. equation (2)], so it is

$$\begin{aligned} \frac{1}{(z-1)(z-2)} &= \frac{1}{z-2} - \frac{1}{z-1} = \sum_{k=0}^{\infty} 2^k z^{-k-1} - \sum_{k=0}^{\infty} z^{-k-1} \\ &= \frac{1}{z^2} + \frac{3}{z^3} + \frac{7}{z^4} + \ldots \end{aligned}$$

NOTE:
These series tell you nothing about the residues of the poles of $\dfrac{1}{(z-1)(z-2)}$ because they are not Laurent Series about the poles $z = 1$ or $z = 2$.

Exercise

We can show that the expansion for $f(z) = \dfrac{1}{(z-1)(z-2)}$ about $z=1$ is $-\dfrac{1}{z-1} - \sum_{k=0}^{\infty}(z-1)^k$ which is valid for $0 < |z-1| < 1$. The coefficient of the $\dfrac{1}{z-1}$ term tells us the residue of $f(z)$ at $z=1$ is -1.

Find the expansion for $f(z) = \dfrac{1}{(z-1)(z-2)}$ about $z=2$ and show that the coefficient of the $\frac{1}{z-2}$ term is 1.

General Formula For Laurent Series of $\dfrac{1}{z-z_0}$ about $z = z_1$

In general, we either have a MacLaurin Series that converges in the region $|z-z_1| < |z_1 - z_0|$ that looks like

$$\begin{aligned}\frac{1}{z-z_0} &= \frac{1}{z-z_1+z_1-z_0} = \frac{1}{z_1-z_0}\frac{1}{1+\frac{z-z_1}{z_1-z_0}} \\ &= \sum_{k=0}^{\infty}(z-z_1)^k(-1)^k(z_1-z_0)^{k-1} \qquad \text{valid when } |z-z_1| < |z_1-z_0| \qquad (4)\end{aligned}$$

or a Laurent Series that converges in the region $|z-z_1| > |z_1-z_0|$ that looks like

$$\begin{aligned}\frac{1}{z-z_0} &= \frac{1}{z-z_1+z_1-z_0} = \frac{1}{z-z_1}\frac{1}{1+\frac{z_1-z_0}{z-z_1}} = \frac{1}{z-z_1}\frac{1}{1-\frac{z_0-z_1}{z-z_1}} \\ &= \sum_{k=0}^{\infty}(z-z_1)^{-k-1}(z_0-z_1)^k \qquad \text{valid when } |z-z_1| > |z_0-z_1| \qquad (5)\end{aligned}$$

Of course we are assuming that $z_1 \neq z_0$ in both these cases because if $z_1 = z_0$ then the expansion is simply $\dfrac{1}{z-z_0}$.

GROUPWORK

Zill & Shanahan, page 282. Expand $f(z) = \dfrac{1}{z(z-1)}$ in a Laurent Series which is valid for the following annular domains.

(a) $0 < |z| < 1$
(b) $1 < |z|$
(c) $0 < |z-1| < 1$
(d) $1 < |z-1|$

RECALL

HINT: $\dfrac{1}{1+\square}$ is equal to $1 - \square + \square^2 - \square^3 + \ldots = \sum_{k=0}^{\infty} (-1)^k \square^k$ when $|\square| < 1$