Complex Analysis

Math 214 Spring 2014 Fowler 307 MWF 3:00pm - 3:55pm

 http://faculty.oxy.edu/ron/math/312/14/

Class 29: Friday April 18

TITLE Linear Fractional Transformations, Continued
CURRENT READING Zill & Shanahan, §7.2
HOMEWORK Zill & Shanahan, §7.2: 14, 15, 21, **27***.

SUMMARY
We shall get more practice finding and using linear fractional transformations.

Linear Fractional Transformations (LFTs)
We continue to consider transformations of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

To determine the equation of an LFT all you have to do is know where it maps 3 distinct points z_1, z_2 and z_3 to w_1, w_2 and w_3, where none of the points z_i and w_i is the point at infinity.
In that case the general form of the LFT is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad \text{(LFT Cross-Ratio Formula)}$$

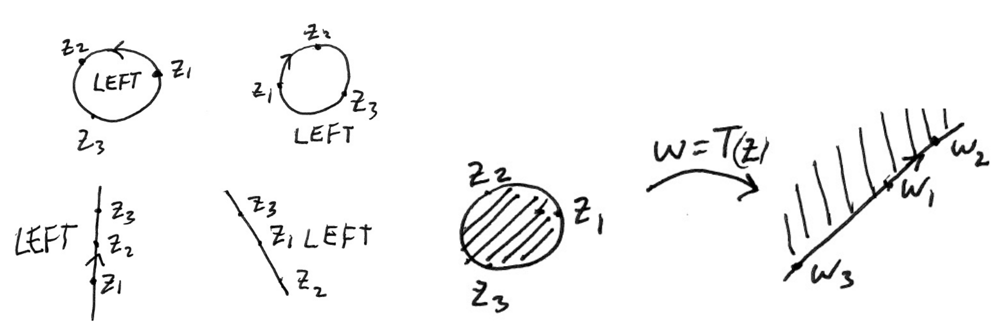
DEFINITION: Cross-Ratio
The cross-ratio $R(z, z_1, z_2, z_3)$ of the complex numbers z, z_1, z_2 and z_3 is defined to be:

$$R(z) = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

THEOREM: **Invariance of Cross-Ratio Under Bilinear Transformation**
Under a linear fractional transformation $w = \dfrac{az+b}{cz+d}$, the cross-ratio $R(z_1, z_2, z_3, z_4)$ becomes $R(w_1, w_2, w_3, w_4)$ but is invariant (unchanged).

$$R(z_1, z_2, z_3, z_4) = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = R(w_1, w_2, w_3, w_4)$$

We can use this preservation property to construct LFT's that do what we want them to do. Given three points on either a circle or a line they will define two regions of space (called the LEFT or RIGHT regions). An LFT acting on these three points will maintain the orientation of the two regions (see diagram below).



EXAMPLE

Suppose we want to map the interior of the circle $|z-2|=2$ so that the points $2-2i \to \frac{i}{2}$, $2+2i \to -\frac{i}{2}$ and $0 \to 0$. What will the image of $|z-2|=2$ be under an LFT? What is the "LEFT" region in this case?

If this is all the information we know about how the function $T(z)$ acts as a mapping, we could also use the LFT Cross-Ratio formula to derive $T(z)$.

Exercise

In fact, we can show that the LFT that maps $2-2i \to \frac{i}{2}$, $2+2i \to -\frac{i}{2}$ and $0 \to 0$ what we want is $w = T(z) = \dfrac{z}{2z-8}$.
(BE CAREFUL: The algebra can become quite challenging!!)

Constructing LFTs With Points At Infinity

Suppose we found out that the previous LFT mapped 4 to ∞.
How do we determine the form of an LFT if one of the points is the point at infinity?
It actually makes the problem *easier*! Suppose we say that our LFT maps z_2 to $w_2 = \infty$.

$\boxed{\text{EXAMPLE}}$

Simply let $w_2 = 1/\beta$ and take the limit as $\beta \to 0$ to show how the formula changes. We can show that the general form of the LFT becomes

$$\begin{aligned}\lim_{\beta\to 0}\frac{(w-w_1)(1/\beta-w_3)}{(w-w_3)(1/\beta-w_1)} &= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\\ \frac{w-w_1}{w-w_3} &= \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\end{aligned}$$

$\boxed{\text{GROUPWORK}}$

Try and write down the linear fraction transformation (LFT), $T(z)$, that maps the points ∞, i and -1 to $1, -i$ and ∞. Think about what are the sets that define the pre-image and the image.

Sketch the pre-image and image under the mapping $T(z)$ that you just found, in the space below.

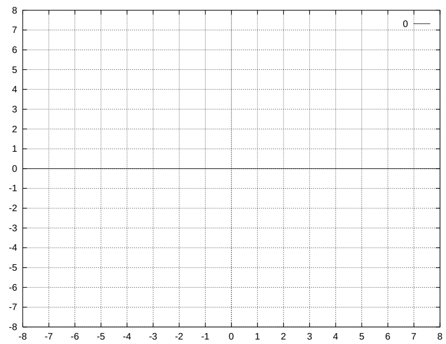
The Most Useful Mapping

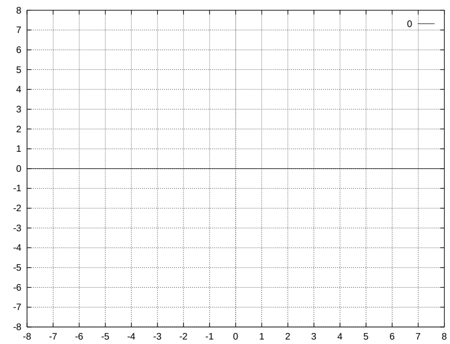
One of the most frequently desired mappings is one which takes the upper-half of the extended complex plane to the interior of the unit circle. Mappings which do this have the following form:

$$w = e^{i\alpha}\frac{z - z_0}{z - \overline{z_0}}, \qquad \text{Im } z_0 > 0, \alpha \text{ Real}$$

Exercise

Draw a sketch illustrating the most useful mapping below by drawing the z-plane and w-plane and give an example of a mapping which would produce the picture you drew.





Representing LFTs Using Matrices

The LFT $w = T(z) = \dfrac{az + b}{cz + d}$ can be represented by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and its inverse T^{-1} can be represented by $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

EXAMPLE

If we know that $w = \dfrac{z - i}{z + i}$ maps the upper half-plane to the interior of the unit circle, find a mapping which maps the interior of the unit circle to the the upper half plane!

Let's confirm that our constructed LFT $w = i\dfrac{1 + z}{1 - z}$ does indeed map the interior of the unit circle to the upper half-plane by checking points.