Complex Analysis

Math 214 Spring 2014 — Fowler 307 MWF 3:00pm - 3:55pm

 — http://faculty.oxy.edu/ron/math/312/14/

Class 23: Monday March 31

TITLE Poles, Zeroes and Residues
CURRENT READING Zill & Shanahan, §6.4-6.5
HOMEWORK Zill & Shanahan, §6.4 2,6,21. §6.5 2,12,17,23.

SUMMARY
We shall be introduced to the concept of residues, and we shall learn about Cauchy's Residue Theorem.

Zeroes and Poles
So, far we have had a lot of experience finding the **poles** of a function and this was important in evaluating contour integrals. The problem of finding a pole is equivalent to finding the **zero** of a related function. Let's formalize these definitions:

DEFINITION: **Zero**
A point z_0 is called a **zero of order** m for the function $f(z)$ if f is analytic at z_0 and f and its first $m-1$ derivatives vanish at z_0, but $f^{(m)}(z_0) \neq 0$.

DEFINITION: **Pole**
A point z_0 is called a **pole of order** m of $f(z)$ if $1/f$ has a zero of order m at z_0.

Identifying Poles and Zeroes
Let f be analytic. Then f has a **zero of order** m at z_0 if and only if $f(z)$ can be written as $f(z) = g(z)(z-z_0)^m$ where g is analytic at z_0 and $g(z_0) \neq 0$.

If $f(z)$ can be written as $f(z) = \dfrac{g(z)}{(z-z_0)^m}$ where $g(z)$ is analytic at z_0, then f has a **pole of order** m at $z = z_0$ and $g(z_0) \neq 0$

How do we find the poles of a function? Well, if we have a quotient function $f(z) = p(z)/q(z)$ where $p(z)$are analytic at z_0 and $p(z_0) \neq 0$ then $f(z)$ has a pole of order m if and only if $q(z)$ has a zero of order m.

EXAMPLE We will classify all the singularities of $f(z) = \dfrac{3z+2}{z^4+z^2}$. How many singularities does $f(z)$ have? And of what order?

GROUPWORK

Let's try and classify all the singularities of the following functions:

(a) $A(z) = \dfrac{4}{z^2(z-1)^3}$

(b) $B(z) = \dfrac{\sin z}{z^2 - 4}$

(c) $C(z) = \tan z$

(d) $D(z) = \dfrac{z}{z^2 - 6z + 10}$

Residues

Once we know all the singularities of a function it is useful to compute the residues of that function. If a function $f(z)$ has a pole of order m at z_0, the residue, denoted by $\mathbf{Res}(f; z_0)$ or $\mathbf{Res}(z_0)$ is given by the formula below:

$$\mathbf{Res}(f; z_0) = \lim_{z \to z_0} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \right\}$$

EXAMPLE

Let's find the residues of the singularities of $f(z) = \dfrac{3z + 2}{z^4 + z^2}$.

Exercise

Find the **residues** of all the singularities we previously classified for the following functions. What is z_0 and m in each case?

(a) $A(z) = \dfrac{4}{z^2(z-1)^3}$

(b) $B(z) = \dfrac{\sin z}{z^2 - 4}$

(c) $C(z) = \tan z$

(d) $D(z) = \dfrac{z}{z^2 - 6z + 10}$

Cauchy's Residue Theorem

If f is analytic on a simple (positively oriented) closed contour Γ and everywhere inside Γ except the finite number of points $z_1, z_2, \cdots z_n$ inside Γ, then

$$\oint_{\Gamma} f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathbf{Res}(f; z_k)$$

EXAMPLE

Let's use the CRT to evaluate the following $\oint_{|z|=2} \frac{3z+2}{z^2(z^2+1)} dz$

sc GroupWork

Use Cauchy's Residue Theorem (CRT) to evaluate the following integrals:

(a) $\oint_{|z|=5} \frac{4}{z^2(z-1)^3}\,dz$

(b) $\oint_{|z|=5\pi} \frac{\sin z}{z^2-4}\,dz$

(c) $\oint_{|z|=2\pi} \tan z\,dz$

(d) $\oint_{|z|=8} \frac{z}{z^2-6z+10}\,dz$