Complex Analysis

Math 214 Spring 2014 | Fowler 307 MWF 3:00pm - 3:55pm

 | http://faculty.oxy.edu/ron/math/312/14/

Class 21: Wednesday March 26

TITLE The Cauchy Integral Formula(s)
CURRENT READING Zill & Shanahan, §5.4-5.5
HOMEWORK Zill & Shanahan, §5.3 2, 9, 12, 20, 25, 27. **23*,29***. §5.4 1, 8, 18, 22. **25***

SUMMARY
We shall be introduced to (one of) the great formulas in complex analysis, if not all of mathematics, Cauchy's Integral Formula, or the CIF.

Cauchy's Integral Formula

Let C be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing C and z_0 is **any point inside** of C then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz$$

The CIF leads to some of the most astonishing results in complex analysis. It is a truly amazing idea; the value of an analytic function at a point z_0 in a simply connected domain depends on values that function takes on some closed contour C encircling the point.

A proof of the result is reasonably straightforward and involves the continuity of $f(z)$ at every point in D and the formula for bounding a contour integral we gave in Worksheet 18. You should try reading the proof on page 234 of Zill & Shanahan and be awestruck by the brilliance of Cauchy!

All Derivatives of Analytic Functions Are Analytic

Here is the first of many amazing ideas derived from the **CIF**.
Let C be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing C and z_0 is **any point inside** of C, then

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz$$

You should be able to write down a general formula for the 2^{nd} derivative of $f(z)$ evaluated at z_0 in terms of a contour integral:

$$f''(z_0) =$$

and in fact we can now write down a general formula for the n^{th} derivative of $f(z)$ evaluated at z_0 in terms of a contour integral:

$$f^{(n)}(z_0) =$$

This is an amazing result, because it means that when a function is analytic then **all** of its higher derivatives exist and are also each analytic!

EXAMPLE

$$\int_{|z|=3} \frac{e^{\pi z}}{(2z+i)(z+2)}\, dz =$$

Exercise

$$\oint_C \overline{z}\, dz, \qquad C : |z| = 2 \text{ clockwise once.}$$

$$\oint_C \frac{dz}{(z-3)^4} \qquad C : |z| = 2 \text{ twice counter-clockwise}$$

$$\oint_C \frac{dz}{(z-3)^4} \qquad C : |z-2| = 2 \text{ twice counter-clockwise}$$

GROUPWORK

Evaluate the following integral (state what result you are using to evaluate each integral)

1. $\oint_C \frac{e^{5z}}{z^2}\,dz \qquad C : |2z-1| = 2$ counter-clockwise

2. $\oint_C \frac{x^2-y^2}{2} + xyi\,dz \qquad C : |z-i| = 2$ counter-clockwise

3. $\oint_C \frac{z}{z^2+\pi^2}dz \qquad C : |z| = 3$ counter-clockwise

4. $\oint_C \frac{\sinh(2z)}{z^2+\pi^2}\,dz \qquad C : |z+i| = 3$ counter-clockwise

Generalized Cauchy Integral Formula

The generalized CIF allows you to evaluate integrands with more complicated denominators and will involve taking the derivative of analytic functions. (Your book calls it Cauchy's Integral Formula for Derivatives).

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz$$

or

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = f^{(n)}(z_0) \frac{2\pi i}{n!}$$

Exercise

Evaluate the following integral along the given contours using the appropriate version of the generalized CIF. Identify $f(z)$, n and z_0 in each case.

$$\oint_C \frac{z+i}{z^3+2z^2} dz$$

where the contour C is

(a) the circle $|z| = 1$ traversed once counter clockwise

(b) the circle $|z+2-i| = 2$ traversed once counter clockwise

(c) the circle $|z-2i| = 1$ traversed once counter clockwise

(d) the circle $|z+1| = 2$ traversed once clockwise