Complex Analysis

Math 214 Spring 2014 — Fowler 307 MWF 3:00pm - 3:55pm

 — http://faculty.oxy.edu/ron/math/312/14/

Class 15: Wednesday February 26

TITLE Complex Exponents z^c and c^z
CURRENT READING Zill & Shanahan, Section 4.2
HOMEWORK Zill & Shanahan, §4.2 # **4**, 9,10,**1**, **13***;

SUMMARY
We have previously looked at roots of complex numbers, i.e. complex numbers being raised to fractional exponents. Let's expand our thinking by considering complex numbers being raised to complex exponents!

Roots of Complex Numbers (Reprise)

From previous identities about complex logarithms and the complex exponential, we can show (for $z \neq 0$) that

$$z^n = \exp(n \log z), \qquad \text{as long as } n \in \mathbb{Z}$$

Similarly,

$$z^{1/n} = \exp\left(\frac{1}{n}\log z\right)$$

But

$$\begin{aligned}
\exp\left(\frac{1}{n}\log z\right) &= \exp\left(\frac{1}{n}[\ln|z| + i(\text{ Arg } z + 2k\pi)]\right) \qquad (k \in \mathbb{Z}) \\
&= |z|^{1/n}\exp\left(i[\frac{\text{Arg } z}{n} + \frac{2k\pi}{n}]\right) \\
&= |z|^{1/n}\exp\left(\frac{i\theta + 2k\pi i}{n}\right) \qquad \text{where } \theta = \text{ Arg } z
\end{aligned}$$

But you should recognize the right hand side as the familiar formula for finding the root of a complex number, where k is restricted to $0, 1, 2, \ldots, n-1$. Why would we do that? [HINT: how many distinct values does $\exp(2k\pi i/n)$ have when k can be any integer and n is fixed?] What do you think happens i f we try and raise a complex number to something besides an integer or rational number? How will we deal with **complex exponents**?

Complex Exponents

If $z \neq 0$ and $c \in \mathbb{C}$, the function z^c is defined as

$$z^c = \exp(\log z^c) = \exp(c \log z)$$

Since $\log z$ is a multi-valued function, z^c will have multiple values. How many values depends on the nature of c.

$$z^c = \begin{cases} z^{n/m} & \text{if } c \text{ is rational, i.e. } n/m & \text{finite number of values (m)} \\ z^n & \text{if } c = n\text{, where } n \text{ is an integer} & \text{single value} \\ z^c & \text{all other complex numbers} & \text{infinite number of values} \end{cases}$$

EXAMPLE

Show that i^i is purely real.

GroupWork

Compute the following:

(a) $(0.5 - \frac{\sqrt{3}}{2}i)^3 =$

(b) $(-1)^{2/3} =$

(c) $(1+i)^{1-i} =$

Derivatives of z^c and c^z

If you choose a branch of z^c which is analytic on an open set, then

$$\frac{d}{dz}(z^c) = cz^{c-1}$$

where the branch of the log used in evaluating z^c is the same branch used in evaluating z^{c-1}
Similarly, we can define the *complex exponential function with base c*

$$c^z = \exp(z \log c)$$

If a single value of c is chosen, then c^z is an entire function such that

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\exp(z \log c) = c^z \log c$$

EXAMPLE

If $f(z) = (1+i)^z$, Find $f'(1-i)$ (HINT: Use Ln)

If $g(z) = z^{(1-i)}$, Find $f'(1+i)$ (HINT: Use Ln)