
Complex Analysis

Math 214 Spring 2014
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Fowler 307 MWF 3:00pm - 3:55pm
<http://faculty.oxy.edu/ron/math/312/14/>

Class 14: Monday February 24

TITLE The Complex Logarithm

CURRENT READING Zill & Shanahan, Section 4.1

HOMEWORK Zill & Shanahan, §4.1.2 # 23,31,34,42 44*;

SUMMARY

We shall return to the murky world of branch cuts as we expand our repertoire of complex functions when we encounter the complex logarithm function.

The Complex Logarithm $\log z$

Let us define $w = \log z$ as the inverse of $z = e^w$.

NOTE

Your textbook (Zill & Shanahan) uses \ln instead of \log and Ln instead of Log .

We know that $\exp[\ln |z| + i(\theta + 2n\pi)] = z$, where $n \in \mathbb{Z}$, from our knowledge of the exponential function.

So we can define

$$\log z = \ln |z| + i \arg z = \ln |z| + i \text{Arg } z + 2n\pi i = \ln r + i\theta$$

where $r = |z|$ as usual, and θ is the argument of z

If we only use the principal value of the argument, then we define the principal value of $\log z$ as $\text{Log } z$, where

$$\text{Log } z = \ln |z| + i \text{Arg } z = \text{Log } |z| + i \text{Arg } z$$

Exercise

Compute $\text{Log } (-2)$ and $\log(-2)$, $\text{Log } (2i)$, and $\log(2i)$, $\text{Log } (-4)$ and $\log(-4)$

Logarithmic Identities

$z = e^{\log z}$ but $\log e^z = z + 2k\pi i$ (Is this a surprise?)

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

However these do not necessarily apply to the principal branch of the logarithm, written as $\text{Log } z$. (i.e. is $\text{Log } (2) + \text{Log } (-2) = \text{Log } (-4)$?)

Log z : the Principal Branch of $\log z$

$\text{Log } z$ is a single-valued function and is analytic in the domain D^* consisting of all points of the complex plane *except for those lying on the nonpositive real axis*, where

$$\frac{d}{dz} \text{Log } z = \frac{1}{z}$$

Sketch the set D^* and convince yourself that it is an open connected set. (Ask yourself: Is every point in the set an interior point?)

The set of points $\{z \in \mathbb{C} : \text{Re } z \leq 0 \cap \text{Im } z = 0\}$ is a line of discontinuities known as a **branch cut**. By putting in a branch cut we say that we “construct $\text{Log } z$ from $\log z$.”

Analyticity of $\text{Log } z$

We can use a version of the Cauchy-Riemann Equations in polar coordinates to help us investigate the analyticity of $\text{Log } z$

Why don't we investigate the analyticity of $\log z$?

If $x = r \cos \theta$ and $y = r \sin \theta$ one can rewrite $f(z) = u(x, y) + iv(x, y)$ into $f = u(r, \theta) + iv(r, \theta)$ in that case, the CREs become:

$$u_r = \frac{1}{r}v_\theta, \quad v_r = -\frac{1}{r}u_\theta$$

and the expression for the derivative $f'(z) = u_x + iv_x$ can be re-written

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

EXAMPLE

Using this information, show that $\text{Log } z$ is analytic and that $\frac{d}{dz} \text{Log } z = \frac{1}{z}$. (HINT: You will need to write down $u(r, \theta)$ and $v(r, \theta)$ for $\text{Log } z$)

Log z As A Mapping Function*Circular Arcs Get Mapped To Vertical Line Segments*

$w = \text{Log } z$ maps the set $\{|z| = r \cap \theta_0 \leq \text{Arg } z \leq \theta_1\}$ onto the vertical line segment $\{\text{Re } (w) = \text{Ln}(r) \cap \theta_0 \leq \text{Im } (w) \leq \theta_1\}$

Segments of Rays Get Mapped To Horizontal Line Segments

$w = \text{Log } z$ maps the set $\{r_0 \leq |z| \leq r_1 \cap \text{Arg } z = \theta\}$ onto the horizontal line segment $\{\text{Ln}(r_0) \leq \text{Re } (w) \leq \text{Ln}(r_1) \cap \text{Im } (w) = \theta\}$

Complex Plane Without Origin Gets Mapped To Infinite Horizontal Strip Of Width 2π

$w = \text{Log } z$ maps the set $|z| > 0$ onto the region $\{-\infty < \text{Re } (w) < \infty \cap -\pi < \text{Im } (w) \leq \pi\}$

GROUPWORK

Find the image of the annulus $2 \leq |z| \leq 2$ under the mapping of the principal logarithm $\text{Ln } z$.

Zill & Shanhan, page 164, #35. Solve the equation $e^{z-1} = -ie^3$