Complex Analysis

Math 214 Spring 2014 — Fowler 307 MWF 3:00pm - 3:55pm

 — http://faculty.oxy.edu/ron/math/312/14/

Class 13: Friday February 21

TITLE The Complex Exponential and Complex Trigonometric Functions
CURRENT READING Zill & Shanahan, Section 4.1 and Section 4.3
HOMEWORK Zill & Shanahan, §4.1.1 #3,4,7,8, **17***; §4.3 #2,9,37 **53***;

SUMMARY
Now that we know something about analytic functions in general and polynomial functions in particular, we need to expand our repertoire of complex elementary functions to includ our old friends exponential, sine and cosine.

The Complex Exponential e^z
The complex version of the exponential function is defined like this:
$e^z = e^{x+iy} = e^x(\cos y + i \sin y)$, where $|e^z| = e^x$ and $\arg(e^z) = y + 2k\pi$, $k = 0, \pm 1, \pm 2, \ldots$
$\arg(e^z) = y + 2k\pi \qquad (k = 0, \pm 1, \pm 2, \ldots)$

GroupWork

Show that $f(z) = e^z$ is an *entire function* and that $f'(z) = e^z$
Take some time (5 minutes) to try and prove this. You will have to answer the questions:

1: What is an entire function?

2: How do you show that a function is analytic?

3: Do the real and complex parts of e^z obey the CRE?

More Properties of e^z

- e^z is never zero
- $e^z = 1 \iff z = 0 + 2\pi k i$
- $e^{z_1} = e^{z_2} \iff z_1 = z_2 + 2k\pi i, \qquad$ where $k \in Z$
- e^z is a periodic function with period $2\pi i$

Mapping Properties of e^z

Consider the action of the function $w = e^z$ on horizontal lines and vertical lines.
What is the image of the line $\text{Re}\,(z) = \alpha$ under $w = e^z$?

What is the image of the line $\text{Im}\,(z) = \beta$ under $w = e^z$?

Fundamental Regions of e^z

A fundamental region of e^z is that set of points in the complex plane which gets mapped to the entire complex plane under the mapping $w = e^z$.

EXAMPLE

Sketch a *fundamental region* for e^z below

Exercise

Show that the image of the first quadrant $\mathcal{S} = \{z \in \mathbb{C} : \ \text{Re}\, z > 0 \ \cap \ \ \text{Im}\, z > 0\}$ under the mapping $w = e^z$ is the region $\{w \in \mathbb{C} : |w| > 1\}$.

Complex trigonometric functions

Once we have a handle on $\exp z$ we can use it to define other functions, most notably $\sin z$ and $\cos z$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

(RECALL: For $x \in \mathbb{R}, \cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$)

EXAMPLE

Show that $\frac{d}{dz} \sin z = \cos z$ by using the definition of $\cos z$ in term of the complex exponential.

There are a whole bunch of typical trigonometric identities which are valid for complex trig functions. Most of these can be proved using the definitions involving exponentials.
For example, $\tan z$ and $\sec z$ are analytic everywhere except at the zeroes of $\cos z$.

Exercise

1. Find the zeroes of $\cos z$ and $\sin z$

2. For what values of z does $\cos(z) = 2$?

3. Show that $\sin(\overline{z}) = \overline{\sin(z)}$

Complex Trigonometric Identities

$$\begin{aligned}
\sin(z+2\pi) &= \sin z, & \cos(z+2\pi) &= \cos z \\
\sin(-z) &= -\sin z, & \cos(-z) &= \cos z \\
\sin^2 z + \cos^2 z &= 1, & \tan^2 z + 1 &= \sec^2 z \\
\sin 2z &= 2\sin z \cos z, & \cos 2z &= \cos^2 - \sin^2 z \\
\sec z &= \frac{1}{\cos z}, & \tan z &= \frac{\sin z}{\cos z} \\
\frac{d}{dz}\tan z = \sec^2 z,\ \frac{d}{dz}\sec z &= \sec z \tan z & \frac{d}{dz}\sin z = \cos z,\ \frac{d}{dz}\sin z &= \cos z
\end{aligned}$$

Similarly the hyperbolic trigonometric functions can be defined using the complex exponential and the newly-defined complex trig functions

$$\text{sinh } z = \frac{e^z - e^{-z}}{2}, \qquad \text{cosh } z = \frac{e^z + e^{-z}}{2}$$

Complex Hyperbolic Trigonometric Identities

$$\begin{aligned}
\sinh z &= -i \sin iz, & \cosh z &= \cos(iz) \\
\frac{d}{dz}\text{sinh } z &= \text{cosh } z, & \frac{d}{dz}\text{cosh } z &= \text{sinh } z
\end{aligned}$$

GroupWork

Show that the mapping $w = \sin(z)$
(a) maps the y-axis one-to-one and onto the v-axis
(b) maps the ray $\{z :\ \text{Arg } z = \pi/2\}$ one-to-one and onto the ray $\{w :\ \text{Re } (w) > 1,\ \text{Im } w = 0\}$

A Very Freaky Function: $w = e^{1/z}$

GroupWork

Saff & Snider, page 117, #25. The behavior of the function $e^{1/z}$ near $z = 0$ is *extremely erratic*. Later (in §6.4 of Zill & Shanahan) we shall classify this point as an **essential singularity**. Show that you can find values of z, all located in the tiny disk $|z| < .001$ where $e^{1/z}$ takes on the values (a) i, (b) -1, (c) 6.02×10^{23} (Avogadro's number) and (d) 1.6×10^{-19} (charge on a single electron, in Coulombs).