Class 6: Monday February 3

TITLE Functions of a Complex Variable
CURRENT READING Zill & Shanahan, Section 2.1-2.3
HOMEWORK Zill & Shanahan, §2.1: #3, 8, 14, 20, 36, **27***; §2.2: 7, 11, 12, 22, **27***

SUMMARY
We expand our exploration of complex variables to start considering functions of a complex variable. One of the first things you'll notice is the immediate geometric interpretations of function evaluation.

Functions of a Complex Variable

Given a set S of complex numbers, a function f is a rule which assigns to each $z \in S$ a complex number w. The set S is known as the **domain of definition** of f. (NOTE: The "domain of definition" is not necessarily a **domain** in the formal mathematical sense of the word we discussed earlier.) The set of all $w \in \mathcal{C}$ given by $w = f(S)$ is called the **image of** S **under** f or the **range of** f and is often denoted S'.
The value $w = f(z)$ can be written as $u + iv$, in other words:

$$w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of two real variables. We can really think of a complex function as two associated functions of two variables. (This is the main reason why *Multivariable Calculus* is the pre-requisite for this class!)

EXAMPLE 1

Write $f(z) = z^2 - z + 2i$ in the form $w = u(x, y) + iv(x, y)$

In addition, given a complex function $w(x, y)$ you can always write it in terms of z, $\overline{z}$ and constants.

Exercise 1

Write $w(x, y) = x^2 + iy^2$ in terms of z and $\overline{z}$

Real Function of a Complex Variable

It is possible to have a function of a complex variable which only produces real values. Some examples of such functions are _______________, _______________ and _______________.

Complex Functions As Mappings

As usual, operations using complex variables have geometric significance.
First, let's get more practice evaluating functions of a complex variable:
Using $f(z) = z^3$, compute

(a) $f(2) =$

(b) $f(\sqrt{2} + i\sqrt{2}) = f(2e^{i\pi/4})$

(c) $f(2i) =$

We can't really graph a complex function, so what we do instead is show what the image of a complex function is on particular sets of points in the complex plane.

Complex Function of a (Single) Real Variable

Suppose $x(t)$ and $y(t)$ are functions of a real variable t. The set of points $\mathcal{D}$ consisting of all points $z(t) = x(t) + iy(t)$ for $a \leq t \leq b$ is called a **parametric curve in the complex plane** or a **complex parametric curve**. The function $z(t)$ is also called the **parametrization** of the curve $\mathcal{D}$ in the plane.

Common Parametric Curves in the Complex Plane

Line Segment (from z_0 to z_1)

$$z(t) = z_0(1-t) + z_1 t, \qquad 0 \leq t \leq 1$$

Ray (emanating from z_0 at angle α to the horizontal axis)

$$z(t) = z_0 + te^{i\alpha}, \qquad 0 \leq t \leq \infty$$

Circle (centered at z_0 with radius r)

$$z(t) = z_0 + re^{it}, \qquad 0 \leq t \leq 2\pi$$

Image of a Parametric Curve under a Complex Mapping

Given a complex function of a complex variable $w = f(z)$ and a complex parametric curve $\mathcal{D}$ represented by $z(t)$ for $a \leq t \leq b$ the function $w(t) = f(z(t))$ for $a \leq t \leq b$ is the parametrization of $\mathcal{D}'$, the image of $\mathcal{D}$.

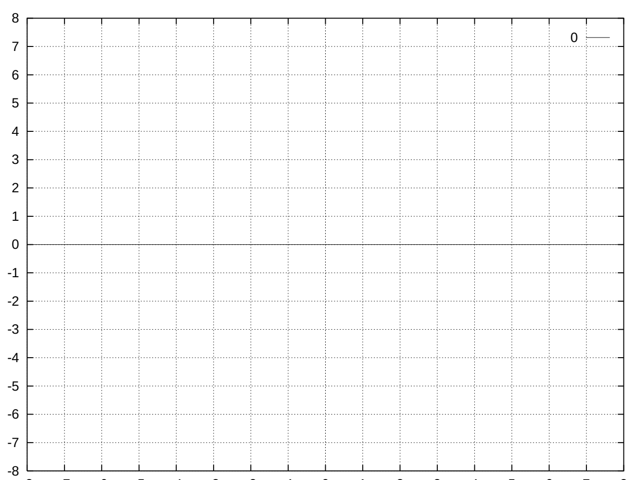
$\boxed{\text{EXAMPLE}}$

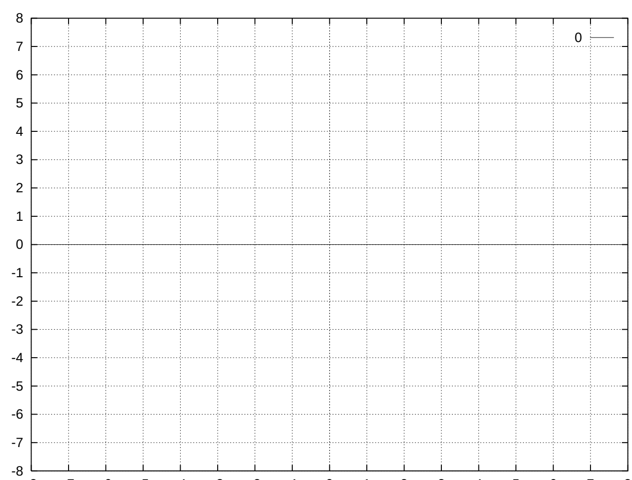
Let's find the image of the line segemnt from 1 to i under the mapping $w = \overline{iz}$.

GroupWork

If you consider $w = f(z)$ a mapping from the z-plane to the w-plane, find the image $\mathcal{D}'$ of $\mathcal{D}$ (the "quarter-disc" of radius 2 centered at the origin) in the first quadrant of the z-plane under the mapping $f(z) = z^3$.

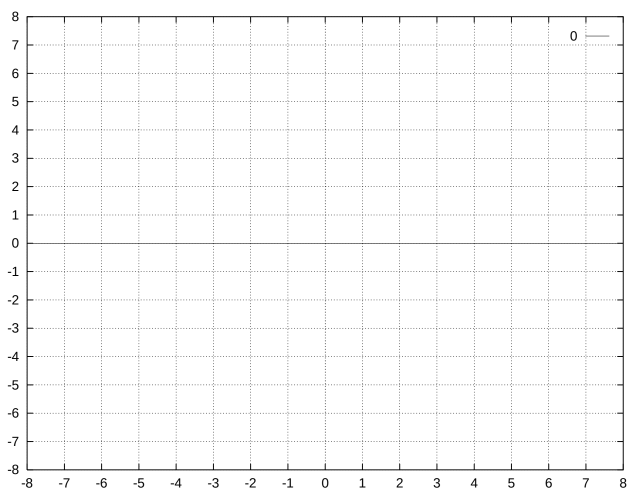
1. Write down a definition of the "quarter disk of radius 2" using complex inequalities
2. Shade in this region on your (x, y) axes (z-plane) below (to the left)
3. Find a parametrization for a ray $\mathcal{A}$ that lies in the quarter-disk.
4. Sketch $\mathcal{A}$ in the z-plane (to the left)
5. Find a parametrization for the image $\mathcal{A}'$ of your curve $\mathcal{D}$ under the mapping $f(z) = z^3$.
6. Sketch $\mathcal{A}'$ in the w-plane (to the right)
7. Use your knowledge of the image of a single curve in the region of interest to determine the image of the entire region. Shade in the mapped region on your (u, v) axes (w-plane) below (to the right)

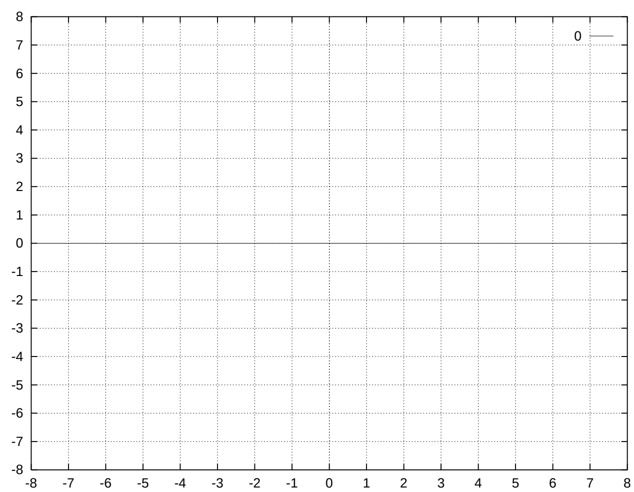




On the following set of axes of axes, do the following:

1. Find a parametrization for a curved arc $\mathcal{B}$ that lies in $\mathcal{D}$.
2. Sketch $\mathcal{B}$ in the z-plane (to the left)
3. Find a parametrization for the image $\mathcal{B}'$ of your curve $\mathcal{D}$ under the mapping $f(z) = z^3$.
4. Sketch $\mathcal{B}'$ in the w-plane (to the right)
5. Use your knowledge of the image of a single curve in the region of interest to determine the image of the entire region of interest under the given mapping. Shade in the mapped region on your (u, v) axes (w-plane) below (to the right)





Exercise

On the following set of axes, sketch the pre-image and the the image of the mapping of the unit "quarter-disk" under $f(z) = 2z^4 - 2 - i$ looks like in the w-plane.