Project for Math 312

Use Contour Integral and Residue Theorem to evaluate real integral

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In this project we will apply the powerful technique of Contour integral in the complex plane to evaluate some improper integrals. These integrals are very difficult to tackle with the regular calculus techniques of real variables. We are going to use the integration along a branch cut, and the residue theorem, plus the proper choice of contours, to solve interesting integrals such as

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = \frac{\pi}{6}$$

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x}dx = \frac{\pi}{sin\alpha\pi}, 0 < \alpha < 1$$

It is quite surprising that by going to one extra dimension, we actually simplify the problem and are able to overcome the seemingly intractable difficulties. This kind of technique is widely used in science and engineering, and clearly shows the elegance and power of complex analysis.

I. Evaluation of $\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)} dx$

To compute Integral $\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)} dx$, we integrate the complex function $f(z) = \frac{1}{\sqrt{z}(z+1)(z+4)}$ along the contour show in figure 6.15 of section 6.6.

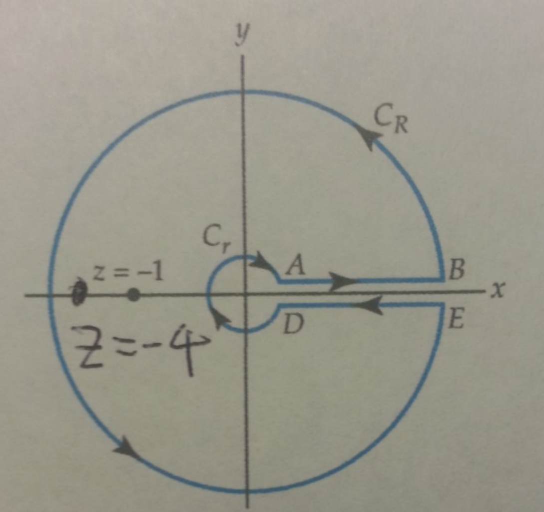

Figure 6.15 Contour for Example 6

The contour consists of four components: C_r and C_R are portions of circles, and *AB* and *ED* are parallel horizontal line segments running along opposite sides of the branch cut. The integrand *f*(*z*) of the contour integral is single valued and analytic on and within *C*, except for the simple poles at *z* = -1 and *z* = -4. Therefore we can write

$$\oint_C \frac{1}{\sqrt{z}(z+1)(z+4)} = 2\pi i(Res(f(z), -1) + Res(f(z), -4))$$

Or $\int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = I_1 + I_2 + I_3 + I_4$

$$= 2\pi i(Res(f(z), -1) + Res(f(z), -4))$$

Along the branch cut, *AB* coincides with the upper side of the positive real axis for which Θ = 0 and *ED* coincides with the lower side of the positive real axis for which Θ = 2π. On *AB*, $= xe^{0i}$, and on *ED*, $z = xe^{2\pi i}$, so that

$$I_2 = \int_{ED} = \int_R^r \frac{\left(xe^{2\pi i}\right)^{-\frac{1}{2}}}{(xe^{2\pi i}+1)(xe^{2\pi i}+4)}\left(e^{2\pi i}dx\right)$$

$$= -\int_R^r \frac{x^{-1/2}}{(x+1)(x+4)}dx = \int_r^R \frac{x^{-1/2}}{(x+1)(x+4)}dx$$

And

$$I_4 = \int_{AB} = \int_r^R \frac{\left(xe^{0i}\right)^{-\frac{1}{2}}}{(xe^{0i}+1)(xe^{0i}+4)}\left(e^{0i}dx\right) = \int_r^R \frac{x^{-1/2}}{(x+1)(x+4)}dx = I_2$$

To evaluate I_1 , let $z = Re^{i\theta}$, we see that

$$|I_1| = |\int_{C_R} \frac{1}{\sqrt{z}(z+1)(z+4)}dz| \leq \frac{2\pi R}{\sqrt{R}(R-1)(R-4)}$$

Since the right hand side -> 0 as $R \to \infty$, we conclude that $\lim_{R\to\infty} I_1 = 0$.

To evaluate I_3 , let $z = re^{i\theta}$, we have

$$|I_3| = |\int_{C_r} \frac{1}{\sqrt{z}(z+1)(z+4)}dz| = |\int_0^{2\pi} \frac{ire^{i\theta}d\theta}{\sqrt{r}e^{i\theta/2}(re^{i\theta}+1)(re^{i\theta}+4)}| \leq \sqrt{r}M$$

where *M* is the upper bound for the analytic function $\frac{1}{(re^{i\theta}+1)(re^{i\theta}+4)}$. Again, we see that $\sqrt{r}M \to 0$ as $r \to 0$, so we can conclude that $\lim_{r\to 0} I_3 = 0$.

Put all these together, we see that

$$\lim_{\substack{R\to\infty \\ r\to 0}} \int_{C_R} + \int_{ED} + \int_{C_r} + \int_{AB} = 2\pi i(Res(f(z), -1) + Res(f(z), -4))$$

Is the same as

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = 2\pi i(Res(f(z), -1) + Res(f(z), -4))$$

What's remaining is to evaluate those 2 residues. Since they are both simple poles, we have

$$Res(f(z),-1) = \frac{1}{\sqrt{z}(z+4)}\bigg|_{z=-1} = \frac{1}{\sqrt{-1}(-1+4)} = \frac{1}{3i}$$

$$Res(f(z),-4) = \frac{1}{\sqrt{z}(z+1)}\bigg| 4 = \frac{1}{\sqrt{-4}(-4+1)} = \frac{1}{-6i}$$

And we get

$$2\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = 2\pi i\left(\frac{1}{3i}+\frac{1}{-6i}\right) = \frac{\pi}{3}$$

Hence finally

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = \frac{\pi}{6}$$

This is different from the stated value in problem 44 of section 6.6, which is $\frac{\pi}{3}$. But I've double checked my calculation and am pretty confident that $\frac{\pi}{6}$ is the correct answer.

Further Discussion: to double check the value of this integral, I decided to use technique of real variable calculus. Use substitution $u = \sqrt{x}, then\ du = \frac{1}{2\sqrt{x}}dx$, so

$$\int_0^\infty \frac{1}{\sqrt{x}(x+1)(x+4)}dx = 2\int_0^\infty \frac{du}{(u^2+1)(u^2+4)} = \frac{2}{3}\int_0^\infty (\frac{1}{u^2+1} - \frac{1}{u^2+4})du$$
$$= \frac{2}{3}(tan^{-1}u - \frac{1}{2}tan^{-1}\frac{u}{2})\,]_0^\infty = \frac{2}{3}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{6}$$

Which is exactly the same value we obtained from complex contour integral!

II. Evaluation of $\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx \,, 0 < \alpha < 1$

To compute Integral $\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx \,, 0 < \alpha < 1$, we integrate the complex function $f(z) = \oint_C \frac{e^{\alpha z}}{1+e^z} dz$ along the rectangular contour show in figure 6.18 of section 6.6:

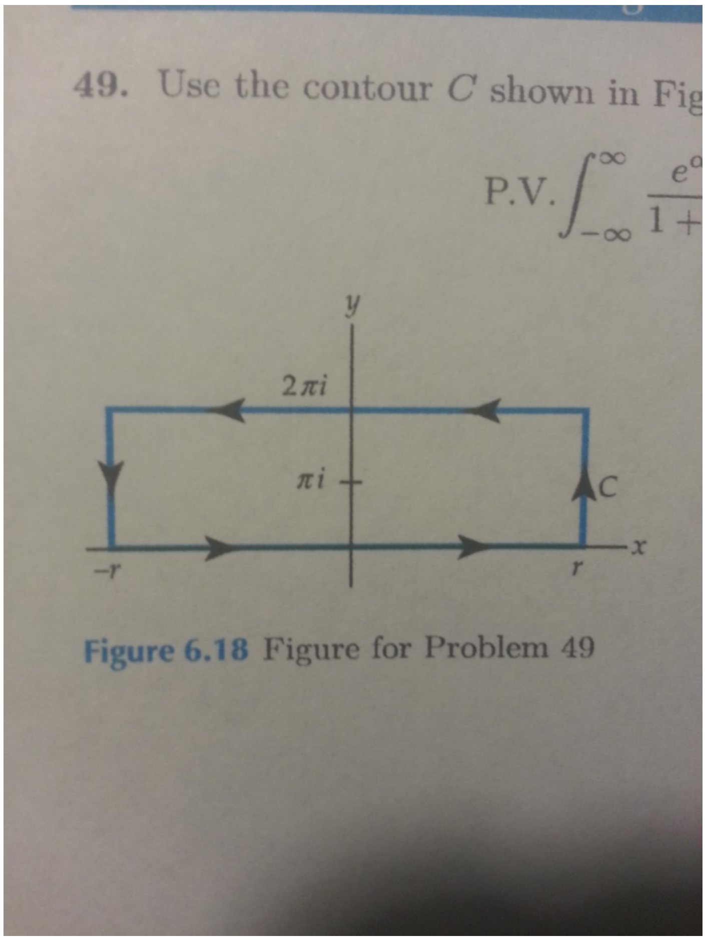

Figure 6.18 Figure for Problem 49

The rectangular contour consists of four components: left and right vertical line segments, plus top and bottom horizontal line segments. The integrand *f*(*z*) of the contour integral is single valued and analytic on and within *C*, except for the simple pole at $z = \pi i$. Therefore we can write

$$\oint_C \frac{e^{\alpha z}}{1+e^z} dz = 2\pi i Res(f(z), \pi i)$$

Or $\int_{top} + \int_{left} + \int_{bottom} + \int_{right} = 2\pi i Res(f(z), \pi i)$

Along the top line segment, $z = x + 2\pi i$, $dz = dx$, so

$$\int_{top} = \int_{r}^{-r} \frac{e^{\alpha(x+2\pi i)}}{1+e^{(x+2\pi i)}} dx = e^{2\alpha\pi i} \int_{r}^{-r} \frac{e^{\alpha x}}{1+e^{x}} dx = -e^{2\alpha\pi i} \int_{-r}^{r} \frac{e^{\alpha x}}{1+e^{x}} dx$$

Along the bottom line segment, $z = x$, $dz = dx$, so

$$\int_{bottom} = \int_{-r}^{r} \frac{e^{\alpha x}}{1+e^{x}} dx$$

Along the left line segment, $z = -r + iy$, $dz = idy$, so

$$\int_{left} = \int_{2\pi}^{0} \frac{e^{\alpha(-r+iy)}}{1+e^{(-r+iy)}} idy$$

We can see that $| \int_{left} | \leq \frac{2\pi e^{-\alpha r}}{1-e^{-r}}$, the right hand side -> 0 as $r \to \infty$, so we can conclude that $\lim_{r\to\infty} \int_{left} = 0$.

Along the right line segment, $z = r + iy$, $dz = idy$, so

$$\int_{right} = \int_{0}^{2\pi} \frac{e^{\alpha(r+iy)}}{1+e^{(r+iy)}} idy$$

We can see that $| \int_{right} | \leq \frac{2\pi e^{\alpha r}}{e^{r}-1}$, the right hand side -> 0 as $r \to \infty$, so we can conclude that $\lim_{r\to\infty} \int_{right} = 0$.

Now put all these 4 line segments together, we arrive at

$$\lim_{r\to\infty} \int_{top} + \int_{left} + \int_{bottom} + \int_{right} = 2\pi i Res(f(z), \pi i)$$

Which is the same as

$$(1 - e^{2\propto\pi i}) \int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^{x}} dx = 2\pi i Res(f(z), \pi i)$$

To compute the residue at $z = \pi i$, since it's a simple pole, we can apply formula (4) of section 6.5:

$$Res(f(z), \pi i) = \left. \frac{e^{\alpha z}}{(1+e^{z})'} \right|_{z=\pi i} = \left. \frac{e^{\alpha z}}{e^{z}} \right|_{z=\pi i} = -e^{\alpha\pi i}$$

So we have

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx = \frac{2\pi i(-e^{\alpha\pi i})}{1-e^{2\propto\pi i}} = \frac{-2\pi i}{e^{-\propto\pi i}-e^{\propto\pi i}} = \frac{-2\pi i}{-2isin\alpha\pi}$$

Which simplifies to

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x}}{1+e^x} dx = \frac{\pi}{sin\alpha\pi}$$

And this is the result we desired.