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Math 312

# Mapping Properties of the Complex Cosine Function

In my opinion, the most fascinating part of complex analysis is the mapping effects of complex functions. However, we have only looked into relatively simple functions, so I decided to investigate the mapping properties of the complex cosine function. In particular, I looked at what effect the complex function $w = \cos(z)$ has on four specific curves: a vertical line, a horizontal line, the unit circle, and a ray at a certain angle to the x-axis. As a guideline for the following calculations, I followed the explanations of the complex sine function in the textbook and on a worksheet online and altered them as needed for the cosine function.

To start out, I manipulated the basic formula for the complex cosine function, $\cos(z) = \frac{1}{2}\,(e^{iz} + e^{-iz})$, to find the real and imaginary parts of $\cos(z)$ as follows: $u = \cos(x)\cosh\,(y)$ and $v = -\sin(x)\sinh\,(y)$ (see calculations).

Looking specifically at a vertical line $x = \alpha$: by letting $x = \alpha$ and further manipulating the real and imaginary parts of the cosine function as given above, I obtained the equation $\frac{u^2}{cos^2\alpha} - \frac{v^2}{sin^2\alpha} = 1$ (see calculations). This is the equation of a hyperbola in the complex plane, meaning the cosine functions maps vertical lines to hyperbolae.

Looking at a horizontal line $y = \beta$: by letting $y = \beta$ and manipulating the real and imaginary parts of $\cos(z)$, I obtained the equation $\frac{u^2}{cosh^2\beta} + \frac{v^2}{sinh^2\beta} = 1$ (see calculations). This is the equation of an ellipse in the complex plane, meaning the cosine function maps horizontal lines to ellipses.

Looking at the unit circle, parameterized by $z(t) = e^{it}, 0 \leq t \leq 2\pi$: By manipulating the original cos $(z)$ function, I found that $\cos(z(t)) = \cos(\cos(t)) \cosh(sin(t)) - i\,(\sin(\cos(t)) \sinh\,(\sin(t))$. By plotting this on Mathematica, I discovered that the unit circle is mapped to a different circle, which is not centered at the origin. Interestingly, if the radius of the original circle is increased, the circular mapping morphs into a shape more reminiscent of a polar graph.

Looking at a ray at angle θ, parameterized by $z(t) = te^{i\theta}, -\infty < t < \infty$: By manipulating the original cos $(z)$ function, I found that $\cos(z(t)) = \cos(t\cos(\theta)) \cosh(t\, sin(\theta)) - i\,(\sin(\mathrm{t}\cos(\theta)) \sinh\,(\mathrm{t}\sin(\theta))$. By plotting this on Mathematica, I saw that the ray is mapped to a strange curve which, as the original angle θ is changed, winds itself around the origin before compressing, flipping over the origin, and unwinding. To see this in action, refer to the Mathematica program included with this project, which allows you to vary the original angle of the ray and see what effect this has on the mapping of the ray by the complex cosine function.

This program also includes graphs for the three previously discussed functions (vertical line, horizontal line, and unit circle). Note that there are two graphs for the unit circle: one is specifically for the unit circle, with the radius of the original circle fixed at one, and the other is for a circle in general, which allows you to change the radius of the original circle and see how the resulting mapping is transformed (as discussed above). The graphs of the vertical and horizontal lines also include a “manipulate” function so you can see how altering $\alpha$ for $x = \alpha$ changes the hyperbola to which the vertical line is mapped or how altering $\beta$ for $y = \beta$ changes the ellipse to which the horizontal line is mapped.

Along with the discussed mapping properties of the cosine function, it should be noted that the cosine function is also an example of a conformal mapping. A conformal mapping is a mapping that preserves angles; in other words, when two angles intersecting at a certain angle in the z-plane are mapped to curves intersecting at the same angle in the w-plane, that mapping is said to be conformal. All complex functions that are analytic are conformal at the points where its derivative is not equal to zero (Zill and Shanahan, 335). Thus, since cosine is an analytic function, it is conformal at all points where its derivative, -sin(z), is not zero (everywhere except multiples of $\pi$). Similarly, sin(z) is an analytic complex function whose derivative, cos(z), is zero at odd multiples of $\frac{\pi}{2}$, so sin(z) is conformal at all other points. This can be seen in the textbook's Table of Conformal Mappings (which does not, unfortunately, include the complex cosine function).

Although there is not a simple way to describe this mapping — like the inversion mapping, which sends lines to circles and circles to lines — it is interesting to see how simple curves are morphed by the complex cosine function to produce completely different curves in the complex plane. For example, it is fascinating that while horizontal and vertical lines are mapped to simple shapes like hyperbolae and ellipses, a straight line at an angle other than 90° is mapped to a curve which behaves in a very strange way. Complex function mappings are really remarkable concepts, and I hope to investigate them more in the future.

## ***Calculations:***

$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$e^{iz} = e^{i(x+iy)} = e^{ix-y} = (\cos(x) + i\sin(x))\, e^{-y}$$

$$e^{-iz} = e^{-i(x+iy)} = e^{-ix+y} = (\cos(x) - i\sin(x))\, e^{y}$$

$$\cos(z) = \frac{1}{2}\left(e^{-y}\cos(x) + i\, e^{-y}\sin(x) + e^{y}\cos(x) - ie^{y}\sin(x)\right)$$

$$= \frac{1}{2}\left(\cos(x)\,(e^{-y} + e^{y}) + i\sin(x)\,(e^{-y} - e^{y})\right)$$

$$= \cos(x)\left(\frac{e^{-y} + e^{y}}{2}\right) + i\sin(x)\,\left(\frac{e^{-y} - e^{y}}{2}\right)$$

$$\cos(x)\cosh(x) - i\sin(x)\sinh(x)$$

$$u = \cos(x)\cosh(x)$$

$$v = -\sin(x)\sinh(x)$$

**<u>Vertical Line $x = \alpha$:</u>**

$$u = \cos(\alpha)\cosh(y) \qquad v = -\sin(\alpha)\sinh(y)$$

$$u^2 = cos^2(\alpha)\, cosh^2(y) \qquad v^2 = sin^2(\alpha) sinh^2(y)$$

$$u^2 = cos^2(\alpha)\,(1 + sinh^2(y)) \qquad sinh^2(y) = \frac{v^2}{sin^2(\alpha)}$$

$$u^2 = cos^2\alpha\left(1 + \frac{v^2}{sin^2(\alpha)}\right)$$

$$\frac{u^2}{cos^2\alpha} = 1 + \frac{v^2}{sin^2(\alpha)}$$

$$1 = \frac{u^2}{cos^2\alpha} - \frac{v^2}{sin^2(\alpha)}$$

**<u>Horizontal Line $y = \beta$</u>**

$$u = \cos(x)\cosh(\beta) \qquad v = -\sin(x)\sinh(\beta)$$

$$u^2 = cos^2(x)\, cosh^2(\beta) \qquad v^2 = sin^2(x) sinh^2(\beta)$$

$$u^2 = (1 - sin^2(x))\, cosh^2(\beta) \qquad sin^2(x) = \frac{v^2}{sinh^2(\beta)}$$

$$u^2 = cosh^2\beta\left(1 - \frac{v^2}{sinh^2(\beta)}\right)$$

$$\frac{u^2}{cosh^2\beta} = 1 - \frac{v^2}{sinh^2(\beta)}$$

$$1 = \frac{u^2}{cosh^2\beta} + \frac{v^2}{sinh^2(\beta)}$$

**Unit Circle: $z(t) = e^{it}$ , $0 \leq t \leq 2\pi$**

$$e^{iz(t)} = e^{i(e^{it})} = e^{i(\cos(t)+i\sin(t))} = e^{i\cos(t)-\sin(t)}$$
$$= (\cos(\cos(t)) + i\sin(\cos(t)))\, e^{-\sin(t)}$$
$$e^{-iz(t)} = e^{-i(e^{it})} = e^{-i(\cos(t)+i\sin(t))} = e^{-i\cos(t)+\sin(t)}$$
$$= (\cos(\cos(t)) - i\sin(\cos(t)))\, e^{\sin(t)}$$
$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\cos(z) = \frac{1}{2}\left(e^{-\sin(t)}\cos(\cos(t)) + ie^{-\sin(t)}\sin(\cos(t)) + e^{\sin(t)}\cos(\cos(t)) - ie^{\sin(t)}\sin(\cos(t))\right)$$
$$\cos(z) = \frac{1}{2}\left[\cos(\cos(t))\left(e^{-\sin(t)} + e^{\sin(t)}\right) + i\sin(\cos(t))\left(e^{-\sin(t)} - e^{\sin(t)}\right)\right]$$
$$\cos(z) = \cos(cos(t))\left(\frac{e^{-\sin(t)} + e^{\sin(t)}}{2}\right) + i\sin(cos(t))\left(\frac{e^{-\sin(t)} - e^{\sin(t)}}{2}\right)$$
$$\cos(z) = \cos(cos(t))\cosh(sin(t)) - i\sin(cos(t))\sinh(sin(t))$$

**Ray at Angle $\theta$ : $te^{i\theta}$ , $-\infty < t < \infty$**

$$e^{iz(t)} = e^{i(t\theta)} = e^{it(\cos(\theta)+i\sin(\theta))} = e^{it\cos(\theta)-t\sin(\theta)}$$
$$= (\cos(t\cos(\theta)) + i\sin(t\cos(\theta)))\, e^{-t\sin(\theta)}$$
$$e^{-iz(t)} = e^{-i(te^{i\theta})} = e^{-it(\cos(\theta)+i\sin(\theta))} = e^{-it\cos(\theta)+t\sin(\theta)}$$
$$= (\cos(t\cos(\theta)) - i\sin(t\cos(\theta)))\, e^{t\sin(\theta)}$$
$$\cos(z) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$
$$\cos(z) = \frac{1}{2}\left(e^{-t\sin(\theta)}\cos(t\cos(\theta)) + ie^{-t\sin(\theta)}\sin(t\cos(\theta)) + e^{t\sin(\theta)}\cos(t\cos(\theta)) - ie^{t\sin(\theta)}\sin(t\cos(\theta))\right)$$
$$\cos(z) = \frac{1}{2}\left[\cos(t\cos(\theta))\left(e^{-t\sin(\theta)} + e^{t\sin(\theta)}\right) + i\sin(t\cos(\theta))\left(e^{-t\sin(\theta)} - e^{t\sin(\theta)}\right)\right]$$
$$\cos(z) = \cos(tcos(\theta))\left(\frac{e^{-t\sin(\theta)} + e^{t\sin(\theta)}}{2}\right) + i\sin(tcos(\theta))\left(\frac{e^{-t\sin(\theta)} - e^{t\sin(\theta)}}{2}\right)$$
$$\cos(z) = \cos(tcos(\theta))\cosh(tsin(\theta)) - i\sin(tcos(\theta))\sinh(tsin(\theta))$$

# Works Cited

"Exponential and Trigonometric Functions." *Hong Kong University of Science and Technology*. N.p., n.d. Web. 15 Apr. 2014. <https://www.math.ust.hk/~maykwok/courses/ma304/06_07/Complex_3.pdf>.

Zill, Dennis G., and Patrick D. Shanahan. *Complex Analysis: A First Course with Applications*. 3rd ed. Burlington: Jones & Bartlett Learning, 2015. Print.