Simplifying Differential Problems using Analytic Mapping

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1 Solving the Dirichlet Problem with Mappings

Problem (taken from example 1 in 4.5)
Let D be the domain in the z-plane bounded by the lines y = x and y = x+2. Find a function ϕ(x,y) that is harmonic in D and satisfies the boundary condition ϕ(x, x+2) = -2 and ϕ(x,x)=3

Solution
This problem will be solved by first mapping it into a domain that is easier to work with preferably one with vertical boundary conditions. Once we obtain this mapping we will then find a solution to the Dirichlet problem and lastly write it in terms of the real and imaginary parts of *f*. By examining the domain of the initial lines a mapping can be found from D to D'

$$f(z) = e^{i\pi/4}z = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)z \tag{1}$$

Using the parameterization for a line z(t) = z_0(1-t)+z_1t, 0$\leq$ t $\leq$ 1 we obtain equations

$$z_1(t) = (1+i)t \tag{2}$$

$$z_2(t) = -2(1-t) + 2it \tag{3}$$

Now plugging in equations 2 and 3 back into equation 1 yields equations 4 and 5 given by

$$z_1(t) = it\sqrt{2} \tag{4}$$

$$z_2(t) = -\sqrt{2} + i(2t\sqrt{2} - \sqrt{2}) \tag{5}$$

where $u_1 = 0$ and $u_2 =$ -$\sqrt{2}$ where $-\infty\leq$ v $\leq\infty$. The boundary conditions ϕ(x,x+2) = -2 and ϕ(x,x) = 3 are transformed to Φ_1(0,v) = 3 and $\Phi_2(-\sqrt{2}$,v) = -2.

The solution to this dirichlet problem was solved on page 190 part (2)

$$\phi(x,y) = \frac{k_1 - k_0}{x_1 - x_0}(x - x_0) + k_0 \tag{6}$$

The benefit of transforming D to D' is because D' has a solution making the task much easier. If we repace x = u, y = v, $x_0 = -\sqrt{2}x_1 = 0$, $k_0 = -2$ and $k_1 = 3$. We obtain

$$\Phi(u,v) = \frac{3-(-2)}{0-\sqrt{2}}(u + \sqrt{2} - 2) = \frac{5}{\sqrt{2}}u + 3 \tag{7}$$

now we need to relate f(z) to Φ(u,v) meaning we need to find the relationship between xy and uv.

$$
\begin{aligned}
f(z) &= \left(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i\right)(x+iy) \qquad (8)\\
&= \frac{\sqrt{2}}{2}(x+y)+i\frac{\sqrt{2}}{2}i(x+y)
\end{aligned}
$$

where u+iv =

$$
u=\frac{\sqrt{2}}{2}(x-y), v=\frac{\sqrt{2}}{2}(x+y)
$$

plugging this into Φ

$$
\begin{aligned}
\phi(x,y) &= \Phi(u[x,y],v[x,y]) \qquad (9)\\
&= \frac{5}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}[x-y]\right)+3 \qquad (10)\\
&= \frac{5}{2}x-\frac{5}{2}y+3 \qquad (11)
\end{aligned}
$$

to test plug in

$$
\phi(x,x)=3=\frac{5}{2}x-\frac{5}{2}x+3=3
$$

$$
\phi(x,x+2)=-2=\frac{5}{2}x-\frac{5}{2}(x+2)+3=-5+3=-2
$$

2 An application of the Dirichlet Problem with Mappings

Problem (taken from problem 14 in 4.5)
Use the anlaytic mapping w = $sin^{-}1$(z) to solve the diriclet problem

Solution

The goal of this problem is to match both curves so that they are much more manageable and easier to deal with as in the example above. One method of doing this is by using the mapping w = $sin^{-}1$(z). When using this process it is easy to take the take the first line and map it however it is much more difficult to map $x^2 - y^2 = 1$. To do this we turn to example 3 of section 4.3. In this example hyperbolas are mapped into vertical lines while ellipses are mapped to horizontal lines. Using this as a basis for our mapping we see that

$$z_1(t) = it$$

is mapped to

$$z_1'(t) = 0$$

and $z_2(t)$ is mapped to

$$z_2'(t) = i2\pi t$$

where $-\infty \leq$ v $\leq \infty$. Now using our known values for ϕ we plug them into equation 6 and obtain the result

$$\phi(x, y) = \frac{-4 - (1)}{2\pi - 0}(x - 0) + 1 = \frac{-5}{2\pi}x + 1 \tag{12}$$

References

[1] Zill, Dennis G. *Complex Analysis* Jones & Bartlett learning.