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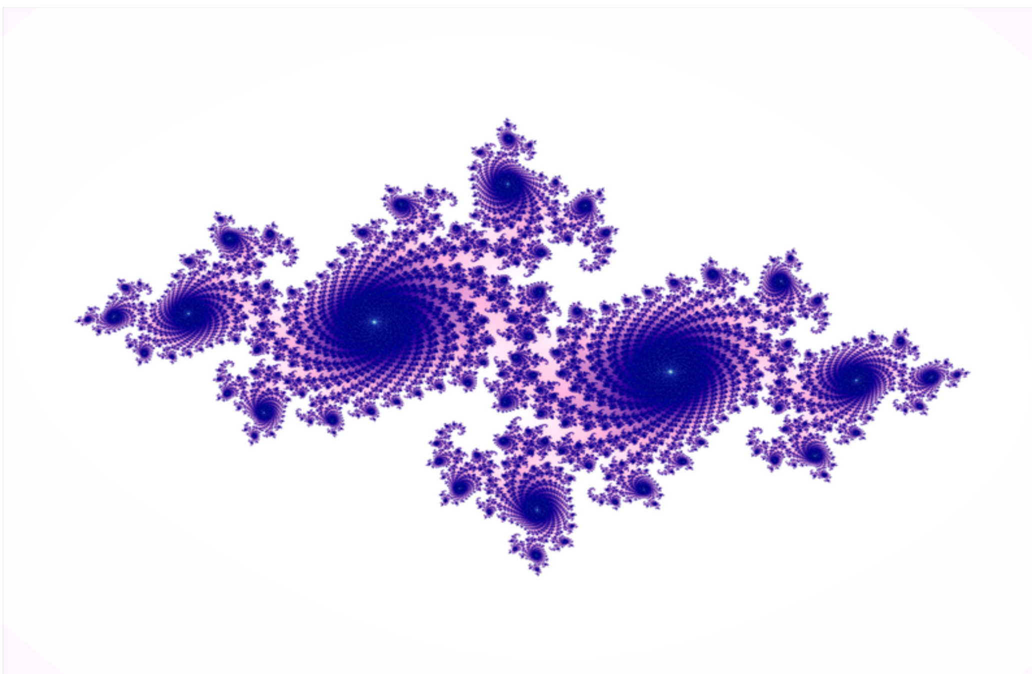
A Discussion of Julia and Mandelbrot Sets

In this paper we will examine the definitions of a Julia Set and the Mandelbrot Set, its characteristics, and the images that they produce. We will explain why there is light and dark shading on Julia set graphs and how iterations are used to produce Julia sets. We will also see the connection between Julia Sets and the Mandelbrot Set. We will present four examples of Julia Sets: two that are connected and two that are disconnected.



Some may look at the image above and marvel at its abstract beauty, but not understand its origin. At first glance it looks like some sort of intense entity you would find in outer space, but there is much more here to be discovered! The picture above is actually a series of fractal images, a detailed mathematical pattern repeating itself over and over again, also known as an iteration. We will define iteration fully later. Fractal images often look like art because of their infinite complexity and eye-appeal.

Specifically, this beautiful image is called a Julia Set. In the earlier 20^{th} century Gaston Julia examined the iterations of polynomials and rational functions, and this set was named after him. His work led to the study of complex dynamics. The formal definition of a Julia Set is as follows:

"Given a rational or transcendental function f, the Julia set of f, denoted J(f) or J, is the closure of the set of repelling and neutral periodic points of f" (Alexander, Lavernaro, and Rosa, 2011).

There are infinitely many Julia Sets, and you can zoom in on the image of the set infinitely many times to discover more beauty!

The Mandelbrot set is the set of all the Julia Sets,

named after the mathematician Benoit Mandelbrot. If the Julia Set is totally disconnected, then it is not in the Mandelbrot Set. Only connected Julia Sets are in the Mandelbrot Set.

A connected set is a path-connected space where any two points can be joined on a graph. Therefore, a disconnected set is where some two points within the set cannot be connected by a path within the set.

The set of repelling points of f is where the limit of the set is infinity, and we will see an example later. Neutral periodic points of f do not converge and will continue to oscillate. There will also be an example of this case later.

You may wonder how to calculate the Julia function to obtain such a magnificent image. We will show the first few steps.

The infinite sequence of iterations on the function f(x) with an initial value say x = a_0 look as follows:

$$a_0,\ a_1 = f(a_0),\ a_2 = f(a_1),\ a_3 = f(a_2),\ \ldots$$

To produce iterations in the Julia Set, we will use the equation

$$z_n = z_{n-1}^{2} + (a+bi)$$

where z is a complex number and a+bi = c is a fixed complex number

Each iteration of the complex numbers (as n changes) will produce a sequence of numbers that does or does not tend to infinity in the limit as n goes to infinity. The complex numbers that result in a sequence that tends towards infinity can be shaded a darker color, while those that do not tend towards infinity can be shaded a lighter color. The resulting image of the Julia Set is remarkable.

Let's randomly choose the point c= -.4+.55i and input it into the function f(Z) to make the function f(Z) = Z^2 - .4+.55i. We'll try the initial point Z_0=0

Z_0= 0
Z_1= $(0)^2$ - .4 + 0.55i = -0.4 + 0.55i
Z_2 = $(-0.4 + 0.55i)^2$ - 0.4 + 0.55i = -0.5425+0.11i
Z_3 = $(-0.5425+0.11i)^2$ -0.4 + 0.55i = -0.1177 + .4306i
Z_4 = -0.5715 + 0.4485i
Z_5 = -0.2744 + 0.0372i
Z_6 = -0.3260 + 0.5295i
Z_7 = -0.5741 + 0.2046i
Z_8 = -0.1122 + 0.3149i
Z_9 = -0.4866 + 0.4792i

These numbers are neutral, periodic points, so they do not tend towards infinity. Therefore, the Z_0=0 series is bounded and would be shaded lighter at this point on the graph. Let's try another different initial point Z_0=1+3i:

Z_1= $(1+3i)^2$-.4+.55i = -8.4+6.55i

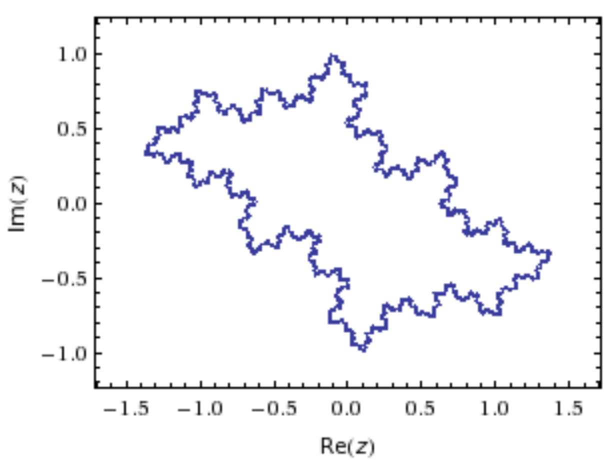
Z_2=(-8.4+6.55i)^2-.4+.55i = 27.2575 + -109.49i
Z_3= (27.2575...-109.49...i)^2-.4+.55i = -11245.5+-5968.29i
Z_4 = 90840444.6 + 134232842i
Z_5 = -9.766*10^{15} + 2.4388*10^{16}i

It appears that for Z_0=1+3i, the series tends towards infinity, so the complex number 1+3i would be shaded darker. Now let's try Z_0=(⅓)+(i/2):

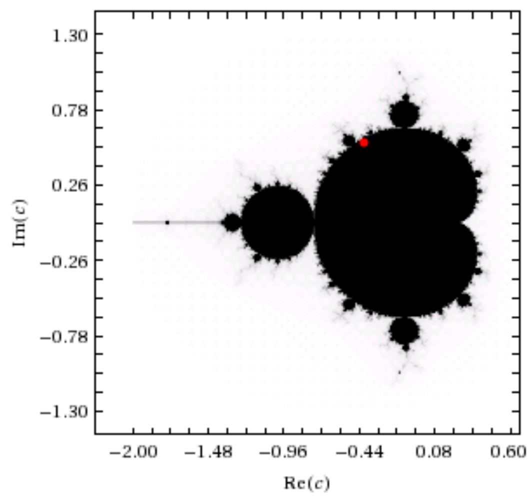
Z_1=((⅓)+(i/2))^2 - 0.4 + 0.55i = -0.5388 + 0.8833i
Z_2=(-0.5388 + 0.8833i)^2-0.4+0.55i = -0.8898 + -0.4020i
Z_3=(-0.8898 - 0.4020i)^2- 0.4+0.55i = 0.2302 +1.2655i
Z_4=(0.2302 + 1.2655i)^2- 0.4+0.55i = -1.9485 +1.1327i

It is unclear if this series is iterating towards infinity or towards the origin, but thankfully, we can use geometry to help us out. The length of the segment from the origin to the point representing (⅓)+(i/2) is .600925 so we will check to see if the distances of Z_{n+1} from the origin are decreasing. The lengths from the origin are Z_1=1.03474, Z_2=.793880, Z_3= 1.28630, and Z_4=2.25389. Although it fluctuates at Z_2, it appears that the points are getting further away from the origin, so we can assume that the series is iterating towards infinity. We also confirm this using an excel spreadsheet. Therefore, the point ((⅓),(i/2)) would be shaded darker.

Fortunately, the computer can do this same process for every complex number to determine what color to shade each of the points. Computing the resulting image from Wolframalpha of the Julia Set is:



From the image we can see that the Julia Set $f(Z) = Z^2 - .4+.55i$ is connected. Because this Julia Set is connected, it is in the Mandelbrot Set. The position of $c=-.4+.55i$ relative to the Mandelbrot Set is depicted below as the red dot.



Now let's try another similar point $c= -.4+.65i$ to get the function $f(Z) = Z^2 - .4+.65i$. We'll try the same initial points $Z_0=0$, $1+3i$, and $(1/3)+(i/2)$. For Z_0 we have:

$Z_0 = 0$
$Z_1 = (0)^2 - 0.4 + 0.65i = -0.4 + 0.65i$
$Z_2 = (-0.4 + 0.65i)^2 - 0.4 + 0.65i = -0.6625 + 0.13i$
$Z_3 = (-0.6625 + 0.13i)^2 -0.4 + 0.65i = 0.0220 + 0.4777i$
$Z_4 = (0.0220 + 0.4777i)^2- 0.4 + 0.65i = -0.6277 + 0.6710i$

We can see that these numbers are neutral, periodic points, so they do not tend towards infinity. Therefore, the $Z_0=0$ series is bounded and would be shaded lighter at this point on the graph. Let's try another different initial point $Z_0=1+3i$:

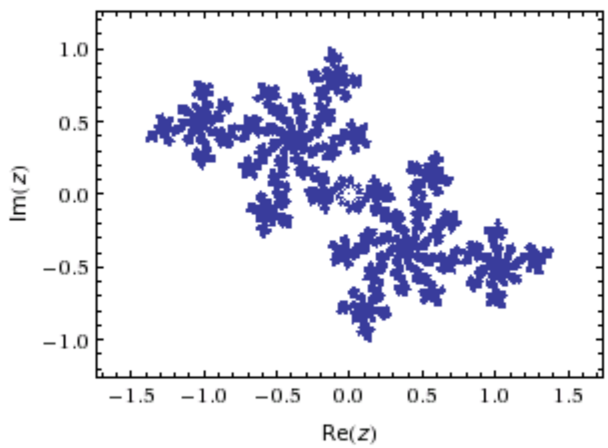
$Z_1= (1 + 3i)^2- 0.4 + 0.65i = -8.4 + 6.55i$
$Z_2=(-8.4 + 6.65i)^2- 0.4 + 0.65i = 25.9375 - 111.07i$
$Z_3= (25.9375 - 111.07i)^2- 0.4 + 0.65i = -11664.191 - 5761.1063i$

It appears that for $Z_0=1+3i$, the series tends towards infinity, so the complex number 1+3i would be shaded darker. Now let's try $Z_0=(⅓)+(i/2)$:

$Z_1=((⅓)+(i/2))^2- 0.4 + 0.65i = -0.5388889 + 0.9833333i$

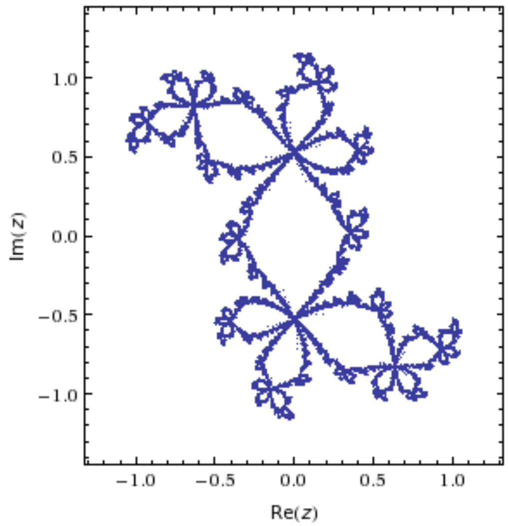
$Z_2=(-0.5388889 + 0.9833333i)^2 - 0.4 + 0.65i = -1.0765432 - 0.4098148i$
$Z_3=(-1.0765432 - 0.4098148i)^2 - 0.4 + 0.65i = 0.5909971 + 1.53236671i$
$Z_4=(0.5909971 + 1.53236671i)^2 - 0.4 + 0.65i = -2.3988702 + 2.46124857i$

It is unclear if this series is iterating towards infinity or towards the origin, but using a spreadsheet in excel we will find that the series iterates towards infinity. Therefore, it will be shaded darker. If we continue this process, we obtain an entirely different image of a Julia Set, even though our c values differ only from 0.1i. Using Wolframalpha to do the rest of our calculations, the Julia set looks like the following:

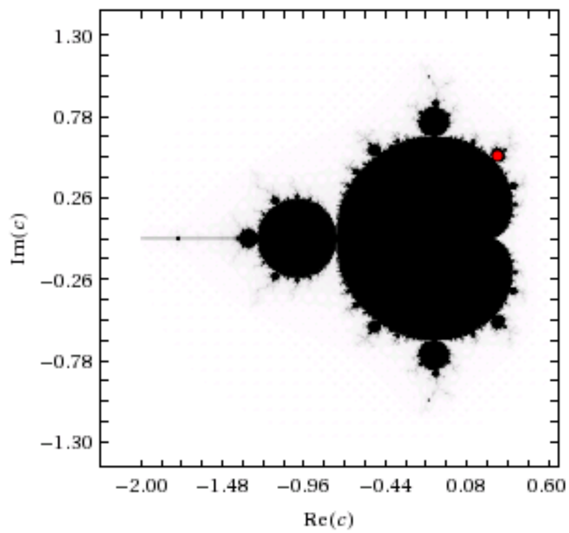


Looking at the image of the set, we can see that the Julia Set is totally disconnected, which means we can assume that there is no Mandelbrot set for c=-.4+.65i.

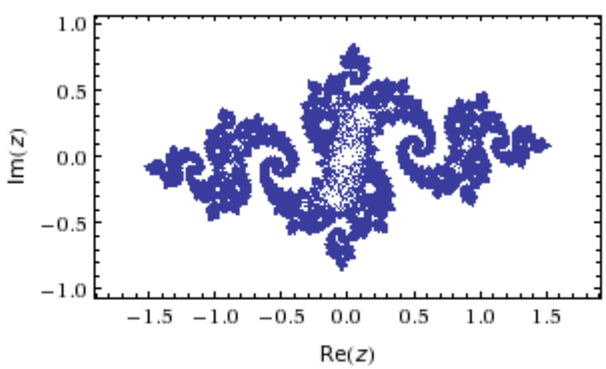
For the Julia Set where c= 0.28+0.528i, we find that the set is connected so it is in the Mandelbrot Set. The image of the Julia set is:



And we see that c = 0.28+0.528i is in the Mandelbrot set as indicated by the red dot.



We observe that the Julia Set where c= -0.8-.156i is totally disconnected.



Therefore, c = -0.8-0.156i does not lie within the Mandelbrot Set.

We have now seen disconnected and connected Julia Sets. There seems to be a correlation between the shading in and the disconnection of a Julia Set. On the other hand, a connected Julia set is just a thin outline graph. We hypothesize that since the shaded Julia Set indicates that every point goes to infinity, and the thin outline Julia Set indicates that the interior points do not go to infinity, the shading of the Julia Set must tie to disconnected and connected sets. This would be a subject of further research and we will make no definitive conclusion within this paper.

In conclusion, we have seen how Julia Sets are created, how to graph them, and their connection to the Mandelbrot Set. We have explored the definitions of connected sets, iterations, and neutral periodic sets.

References

Alexander, Lavernaro, and Rosa. *Early Days in Complex Dynamics: A History of Complex Dynamics in One Variable During 1906-1942.* London Mathematical Society. 2011.

Joyce, David. *Julia and Mandelbrot Sets*. Last updated: May, 2003. <http://aleph0.clarku.edu/~djoyce/julia/julia.html> Date accessed: April 29th, 2014.

Schovanec, Valerie. *Complex Numbers and Imaginary Fractals: Julia Sets*. 2010. <http://scimath.unl.edu/MIM/files/MATExamFiles/Schovanec_Final%20072610_feedback%205.pdf> Date accessed: April 29th, 2014.

<http://en.wikipedia.org/wiki/Julia_set> Date accessed: April 29th, 2014.

<http://www.wolframalpha.com/> Date accessed: April 29th, 2014.