Complex Analysis

Math 214 Spring 2004 Fowler 316 MWF 3:30pm - 4:25pm
 http://faculty.oxy.edu/ron/math/312/04/

Class 34: **Monday April 19**

SUMMARY Linear fractional transformations and mappings
CURRENT READING Saff & Snider, §7.5

Linear Fractional Transformations (LFTs)
We continue to consider transformations of the form

$$w = T(z) = \frac{az+b}{cz+d}$$

Previously we showed that it took us evaluating $T(z)$ at 3 different points for us to determine exactly what the shape of the mapped image would look like.
In fact, to determine the equation of an LFT all you have to do is know where it maps 3 distinct points z_1, z_2 and z_3 to w_1, w_2 and w_3, where none of the points z_i and w_i is the point at infinity.
In that case the general form of the LFT is

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

EXAMPLE
Remember that previously we have dealt with an LFT which mapped the interior of $|z-2| = 2$ to the negative half of the complex plane.
If this is all the information we know about how the function $T(z)$ acts as a mapping, we could use the above formula to derive $T(z)$

We saw that it mapped $0, 2+2i$ and 2 to $0, -i/2$ and $-1/2$ (respectively).
Using the formula above, show that the mapping that does this is $w = T(z) = \dfrac{z}{2z-8}$
(CAREFUL: The algebra can get pretty tricky!)

Remember that we found out that the previous LFT mapped 4 to ∞.
How do we determine the form of an LFT if one of the points is the point at infinity?
It actually makes the problem *easier*! Suppose we say that our LFT maps z_2 to $w_2 = \infty$.

EXAMPLE

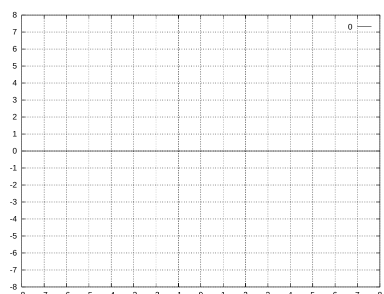
Simply let $w_2 = 1/\beta$ and take the limit as $\beta \to 0$ to show how the formula changes. We can show that the general form of the LFT becomes

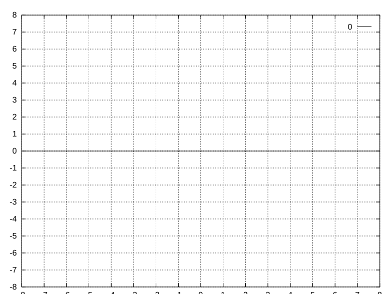
$$\lim_{\beta\to 0} \frac{(w - w_1)(1/\beta - w_3)}{(w - w_3)(1/\beta - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

GROUPWORK

Try and write down the linear fraction transformation (LFT), $T(z)$, that maps the points ∞, i and -1 to $1, -i$ and ∞

Sketch the pre-image and image under the mapping $T(z)$ that you just found, on the axes below.





The Most Useful Mapping

One of the most frequently desired mappings is one which takes the upper-half of the extended complex plane to the interior of the unit circle. Mappings which do this have the following form:

$$w = e^{i\alpha}\frac{z - z_0}{z - \overline{z_0}}, \qquad \text{Im } z_0 > 0, \alpha \text{ Real}$$

Draw a sketch illustrating the mapping below by drawing the z-plane and w-plane and give an example of a mapping which would produce the picture you drew.