Complex Analysis

Math 214 Spring 2004 — Fowler 316 MWF 3:30pm - 4:25pm

 — http://faculty.oxy.edu/ron/math/312/04/

Class 32: **Wednesday April 14**

SUMMARY Cauchy Principal Value of an Improper Integral of the First Kind
CURRENT READING Saff & Snider, §6.3

Exercise

Consider the improper integral $\int_{-\infty}^{\infty} x^3\, dx$

What do you have to do before you can evaluate the integral?

Is this the same value as $\lim_{R\to\infty} \int_{-R}^{R} x^3\, dx$?

Cauchy Principal Value

The Cauchy Principal Value of an improper integral, denoted by p.v.$\int_{-\infty}^{\infty} f(x)\, dx$ is defined as

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$$

GROUPWORK

Find p.v.$\int_{-\infty}^{\infty} x\, dx$ and p.v.$\int_{-\infty}^{\infty} x^2\, dx$

Compare these answers to the improper integral $\int_{-\infty}^{\infty} x\, dx$ and $\int_{-\infty}^{\infty} x^2\, dx$

What's the difference between two sets of answers? Notice any patterns?

The relationship between the Cauchy Principal Value of an improper integral and the improper integral can be sumarized as

convergence of $\int_{-\infty}^{\infty} f(x)\ dx$ IMPLIES p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ EXISTS

p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ EXISTS DOES NOT IMPLY convergence of $\int_{-\infty}^{\infty} f(x)\ dx$

There is a condition on $f(x)$ from which we will know when the two values are equal: If $f(x)$ is an EVEN FUNCTION or if the improper integral converges.

p.v. $\int_{-\infty}^{\infty} f(x)\ dx$ IS EQUAL TO $\int_{-\infty}^{\infty} f(x)\ dx,$ when $f(x) = f(-x)$

Evaluation of real integrals of the form $\int_{-\infty}^{\infty} f(x)\ dx$ using Residues

We can also evaluate **Improper Integrals** more easily by evaluating associated contour integrals. however, we have to have some conditions on the integrand $f(z)$.

You can use some boundedness theorems to say that

$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)\ dz = \oint_C f(z)\ dz$$

and then take the limit as $R \to \infty$ to say that

$$\int_{-\infty}^{\infty} f(x)\ dx = \oint f(z)\ dz = 2\pi i \sum Res(f)$$

if the boundedness theorems above apply to $f(z)$ (since then $\int_{C_R} f(z)\ dz \to 0$ as $R \to \infty$)

EXAMPLE

Show that $\int_0^{\infty} \frac{1}{x^4+1}\ dx = \frac{\sqrt{2}}{4}\pi$

Exercise

Find the Cauchy Principal Value of $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ by evaluating a related contour integral for a function $f(z)$.

(Why do you know you can use this method?)

(Is the value of the integral the same as the cauchy principal value? Why/why not?)

Evaluating Improper Integrals using Jordan's Lemma

$$\text{p.v.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx = \oint f(z)\ dz = 2\pi i \sum Res(f)$$

As long as $f(z)$ obeys the two boundedness theorems such that $|f(z)| < M/|z|^k$ where $k > 1$. (In other words if $f(z)$ is a rational function $p(z)/q(z)$ then the degree of $q(z)$ must be greater than degree of $p(z) + 1$.)

Boundedness Theorem 1

If $|f(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$ and M are constants and C_R is the closed contour consisting of the real axis from $-R$ to $+R$ together with the semi-circle of radius R from $\theta = 0$ to $\theta = \pi$, then

$$\lim_{R\to\infty} \oint_{C_R} f(z)\ dz = 0$$

Boundedness Theorem 2

If $|f(z)| \leq \dfrac{M}{R^k}$ for $z = Re^{i\theta}$ where $k > 1$, $n > 0$ and M are constants and C is the closed contour consisting of the real axis from $-R$ to $+R$ together with the semi-circle of radius R from $\theta = 0$ to $\theta = \pi$, then

$$\lim_{R\to\infty} \oint_{C} f(z)e^{inz}\ dz = 0$$

The second boundedness theorem is sometimes called **Jordan's Lemma**.

GROUPWORK

Show that (for $a > 0$) $\displaystyle\int_0^{\infty} \frac{\cos(ax)}{x^2+1}\ dx = \frac{\pi}{2}e^{-a}$