Complex Analysis

Math 214 Spring 2004 Fowler 316 MWF 3:30pm - 4:25pm
 http://faculty.oxy.edu/ron/math/312/04/

Class 25: **Monday March 29**

SUMMARY Cauchy's Integral Formula and Integral Examples
CURRENT READING Saff & Snider, §4.5

Cauchy's Integral Formula

Let C be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing C and z_0 is **any point inside** of C then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}dz$$

The CIF leads to some of the most astonishing results in complex analysis. It is a truly amazing idea; that the value of an analytic function at a point z_0 in a simply-connected domain depends on values it takes on some closed contour C encircling the point.

An alternative proof of the result is reasonably straightforward and involves the continuity of $f(z)$ at every point in D and the formula for bounding a contour integral. You might try reading it on page 204-205 of Saff & Snider.

Higher Derivatives of Analytic Functions

Here is the first of many amazing ideas derived from the **CIF**.
Let C be a simple closed (positively oriented) contour. If f is analytic in some simply connected domain D containing C and z_0 is **any point inside** of C, then

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z - z_0)^2}dz$$

and in fact you should be able to write down a general formula for the n^{th} derivative of $f(z)$ evaluated at z_0 in terms of a contour integral:

$$f^{(n)}(z_0) =$$

This is an amazing result, because it means that when a function is analytic then all of its higher derivatives exist and are also each analytic!

EXAMPLE

$$\int_{|z|=3} \frac{e^{\pi z}}{(2z+i)(z+2)}\,dz =$$

GROUPWORK

Sketch the contour and evaluate the following integrals. **1.** $\oint_C \overline{z}\, dz,\qquad C: |z| = 2$ clockwise

2. $\oint_C \frac{x^2 - y^2}{2} + xyi\, dz \qquad C: |z - i| = 2$ counter-clockwise

3. $\oint_C \frac{dz}{(z-3)^4} \qquad C: |z - 2| = 2$ twice counter-clockwise

4. $\oint_C \frac{dz}{z^2 + \pi^2} \qquad C: |z| = 3$ counter-clockwise

5. $\oint_C \frac{\sinh(2z)}{z^2 + \pi^2}\, dz \qquad C: |z + i| = 3$ counter-clockwise

Exercise

Evaluate the following integral

$$\oint_C \frac{z + i}{z^3 + 2z^2} dz$$

where the contour C is

(a) the circle $|z| = 1$ traversed once counter clockwise

(b) the circle $|z + 2 - i| = 2$ traversed once counter clockwise

(c) the circle $|z - 2i| = 1$ traversed once counter clockwise

(d) the circle $|z + 1| = 2$ traversed once clockwise