Complex Analysis

Math 214 Spring 2004 Fowler 112 MWF 3:30pm - 4:25pm
 http://faculty.oxy.edu/ron/math/312/04/

Class 14: **Monday February 23**

SUMMARY The Complex Logarithm
CURRENT READING Saff & Snider, §3.3
HOMEWORK Saff & Snider, Section 3.3 # 1, 2, 3, 4, 5

The Complex Logarithm $\log z$
Let us define $w = \log z$ as the inverse of $z = e^w$
But we know that $\exp[\ln|z| + i(\theta + 2n\pi)] = z$, where $n \in Z$, from our knowledge of the exponential function.
So we can define

$$\log z = \ln|z| + i \text{ arg } z = \ln|z| + i \text{ Arg } z + 2n\pi i = \ln r + i\theta$$

where $r = |z|$ as usual, and θ is the argument of z
If we only use the principal value of the argument, then we define the principal value of $\log z$ as $\text{Log } z$, where

$$\text{Log } z = \ln|z| + i \text{ Arg } z = \text{Log } |z| + i \text{ Arg } z$$

Exercise
Compute $\text{Log }(-2)$ and $\log(-2)$, $\text{Log }(2i)$, and $\log(2i)$, $\text{Log }(-4)$ and $\log(-4)$

Logarithmic Identities
$z = e^{\log z}$ but $\log e^z = z + 2k\pi i$ (Is this a surprise?)
$\log(z_1 z_2) = \log z_1 + \log z_2$
$\log\left(\dfrac{z_1}{z_2}\right) = \log z_1 - \log z_2$
However these do not neccessarily apply to the principal branch of the logarithm, written as $\text{Log } z$. (i.e. is $\text{Log }(2) + \text{Log }(-2) = \text{Log }(-4)$?

Log z: the Principal Branch of $\log z$

Log z is a single-valued function and is analytic in the domain D^* consisting of all points of the complex plane *except for those lying on the nonpositive real axis*, where

$$\frac{d}{dz} \text{ Log } z = \frac{1}{z}$$

Sketch the set D^* and convince yourself that it is an open connected set.
(Ask yourself: Is every point in the set an interior point?)

The set of points $\{ \text{ Re } z \leq 0 \cap \text{ Im } z = 0\}$ is a line of discontinuities known as a **branch cut**. By putting in a branch cut we say that we "construct Log z from $\log z$." Why can we not evaluate $\log z$ along the entire positive x-axis?

Analyticity of Log z

We can use a version of the Cauchy-Riemann Equations in polar coordinates to help us investigate the analyticity of Log z

Why don't we investigate the analyticity of $\log z$?

If $x = r\cos\theta$ and $y = r\sin\theta$ one can rewrite $f(z) = u(x,y)+iv(x,y)$ into $f = u(r,\theta)+iv(r,\theta)$ in that case, the CREs become:

$$u_r = \frac{1}{r}v_\theta, \qquad v_r = -\frac{1}{r}u_\theta$$

and the expression for the derivative $f'(z) = u_x + iv_x$ can be re-written

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$

EXAMPLE

Using this information, show that Log z is analytic and that $\frac{d}{dz}$ Log $z = \frac{1}{z}$.
(HINT: You will need to write down $u(r,\theta)$ and $v(r,\theta)$ for Log z)